Homology of Coxeter and Artin groups

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Declaration

I declare that this thesis has been composed entirely by myself, and that the work contained within is my own. Quotation marks have been used to indicate sources of information and have been specifically acknowledged. I also declare that this thesis has not been accepted in any previous application for a degree.

Signed:

Abstract

We calculate the second and third integral homology of arbitrary finite rank Coxeter groups. The first of these calculations refines a theorem of Howlett, the second is entirely new. We then prove that families of Artin monoids, which have the braid monoid as a submonoid, satisfy homological stability. When the $K(\pi, 1)$ conjecture holds this gives a homological stability result for the associated families of Artin groups. In particular, we recover a classic result of Arnol'd.

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Finally I would like to thank my partner Harry for being by my side for the past seven years, and riding this roller-coaster with me.

Introduction

This thesis is concerned with the homology of Coxeter and Artin groups. Broadly, the thesis can be separated into two parts: the first two chapters cover results that give formulas for the second and third integral homology of a finite rank Coxeter group, and the remaining chapters focus on a homological stability results for families of Artin monoids.

Introduction to Coxeter and Artin groups

Harold Scott MacDonald Coxeter (known as Donald) was one of the greatest geometers of the twentieth century. Born in 1907, son to a sculptor and a painter, he was drawn to geometric shapes as a child, and later to a chapter on 'platonic solids' in his school textbook. Pursuing this interest, he won a prize for an essay on "Dimensional Analogy", and Bertrand Russell, who was friends with his father, read the essay and persuaded Coxeter to pursue mathematics, despite being at the bottom of his class. His continuing fascination with polytopes and geometry led him to rigorously define *regular polytopes*, extending the notion of regular polygons and polyhedra to tessellations, such as honeycombs, and higher dimensional polytopes. The renewed interest in polytope reflection groups in the twentieth century was partially due to the discovery that many polyhedra occur naturally, inherent in crystalline structures. Due to the symmetrical laws of nature, it is the regular polyhedra which occur. However as Coxeter writes:

> "Thus the chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense."
> H.S.M. Coxeter *Regular Polytopes* [**16**, p.vi]

and so it is possible that he required no application, only inherent beauty, to study these objects. Coxeter introduced the symmetry groups of regular polytopes, *Kaleidoscopic groups*, in his 1947 book *Regular Polytopes* [**16**], reviewed in the 1949 Bulletin of the American Mathematical Society:

> "The serious mathematics begins with the third chapter in which Coxeter introduces the symmetry groups of the Platonic solids. After a full discussion of this important topic, he turns to degenerate polyhedra such as tessellations and honeycombs and their groups. These lead to results of crystallographic

> importance. Under the heading "The Kaleidoscope" he then describes the discrete groups generated by reflections. The exposition is greatly illuminated by his own "graphical notation" which makes complicated relations self-evident."
> C. B. Allendoerfer *Bulletin of the AMS 1949* [**3**]

In 1961 Tits introduced the abstract definition of *Coxeter groups* in his preprint *Groupes et geometries de Coxeter*: a Coxeter group is generated by a set of involutions, which satisfy generalised braiding relations [**44**]. One of the most primitive examples of a Coxeter group is the symmetric group on n letters, S_n. The groups of "The Kaleidoscope" were exactly the finite examples of Coxeter groups and the "graphical notation" of Coxeter became known as *Coxeter-Dynkin diagrams.* Coxeter groups play a significant role in many areas of mathematics and they often arise as the foundations of various structures. For example they arise as root systems and indexing sets for Iwahori-Hecke algebras in the representation theory of groups of Lie type, they arise as Weyl groups of Lie algebras and algebraic groups [**27**]. In both geometric and combinatorial group theory, Coxeter groups arise as a rich source of examples, and Tits originally defined Coxeter groups as a stepping stone to developing the theory of buildings [**17**]. Key texts in the study of Coxeter groups, and of particular relevance to this thesis are *The Geometry and Topology of Coxeter Groups* by Davis [**17**], *Reflection Groups and Coxeter Groups* by Humphreys [**33**] and *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras* by Geck and Pfeiffer [**27**].

For every Coxeter group there is a related *Artin group*, where the condition that the generators are involutions is discarded. The Artin group related to the Coxeter group S_n is the *braid ground on n-strands*, $\mathcal{B}_n$. Braids were initially studied in the context of being non-intersecting closed curves in 3-space (for example, see [**2**]), but in 1925 Artin introduced many results on $\mathcal{B}_n$, including the standard presentation. His motivation was to better understand the theory of knots and links. As Joan Birman writes :

> "It is a tribute to Artin's extraordinary insight as a mathematician that the definition he proposed in 1925 for equivalence of geometric braids could ultimately be broadened and generalised in many different directions without destroying the essential features of the theory."
> Joan Birman *Braids, links and mapping class groups* [**7**, p.3]

and indeed this theory was generalized in many ways i.e. to theory of algebraic functions and algebraic equations [**29**], to theory of knots and links [**28**] and to monodromy theory in various forms (for example symplectic monodromy [**5**]). In 1962 Fox and Neuwirth showed that the braid group arose as the fundamental group of configurations of n-points on the plane [**24**]. This can be rephrased as the fundamental group of a quotient of a *hyperplane complement* by the symmetric group S_n, and in 1971 Artin groups were first introduced by Brieskorn [**9**] as the fundamental groups of the quotient of certain hyperplane complements by corresponding Coxeter groups. Brieskorn was motivated by the result of Fox and Neuwirth, alongside conjectures of Tits which speculated that a generalisation of Braid groups in the sense of hyperplane complements should correspond to the Coxeter groups. His main interest was the geometric meaning that these groups had, in terms of singularity theory.

Alongside $\mathcal{B}_n$, the free group and the free Abelian group are also examples of Artin groups. Artin groups can be split into two families: the *finite type* Artin groups are Artin groups associated to finite Coxeter groups, and the *infinite type* Artin groups are Artin groups associated to infinite Coxeter groups. While many results are known in general for finite type Artin groups, much is yet to be determined for infinite type Artin groups. There are many conjectures concerning infinite type Artin groups and one key conjecture in this area is the $K(\pi, 1)$ *conjecture*. This conjecture states that the defining hyperplane complements are in fact classifying spaces for the related Artin groups. A discussion of Artin groups and in particular the $K(\pi, 1)$ conjecture is recorded in Paris's notes on the $K(\pi, 1)$ *conjecture for Artin groups* [**40**].

Results: Low dimensional homology of Coxeter groups

Define $\pi(a, b; k)$ to be a word of length k, given by the alternating product of a and b, i.e.

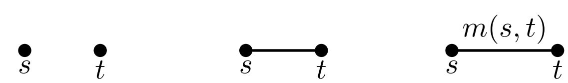
$$\pi(a, b; k) = \overbrace{abab\ldots}^{\text{length k}}.$$

Given a finite generating set S, a Coxeter group W has the following presentation

$$W = \left\langle S \;\middle|\; \begin{array}{cc} s^2 = e & \forall s \in S \\ \pi(s, t; m(s, t)) = \pi(t, s; m(s, t)) & \forall s, t \in S \end{array} \right\rangle$$

where $m(s, t) = m(t, s)$ and $m(s, t)$ is either an integer greater than or equal to 2, or ∞. We call $|S|$ the *rank* of W.

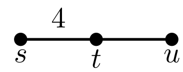
One can package the information given in the presentation of a Coxeter group W into a diagram called a *Coxeter diagram*, denoted $\mathcal{D}_W$. It is the graph with vertices indexed by the elements of the generating set S. Edges are drawn between the vertices corresponding to s and t in S when $m(s, t) \geq 3$ and labelled with $m(s, t)$ when $m(s, t) \geq 4$, as shown below:



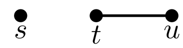
$m(s, t) = 2$ $\quad$ $m(s, t) = 3$ $\quad$ $m(s, t) \geq 4$

In this thesis, variations on this diagram are defined, and Theorems A and B calculate the second and third integral homology for any finitely generated Coxeter group W, in terms of simplicial homologies of these new diagrams. The first theorem is a refinement of a theorem of Howlett [**32**], who computed the rank of the Schur multiplier of a finite rank Coxeter group in 1988. To state this theorem we introduce three new diagrams $\mathcal{D}_{odd}$, $\mathcal{D}_{even}$ and $\mathcal{D}_{\bullet\bullet}$.

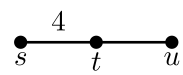
- $\mathcal{D}_{odd}$ is the diagram with vertex set S and an edge between s and t in S if $m(s,t)$ is odd. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



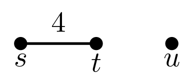
then $\mathcal{D}_{odd}$ is given by



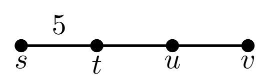
- $\mathcal{D}_{even}$ is the diagram with vertex set S and an edge between s and t in S if $m(s,t)$ is even and not equal to 2. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



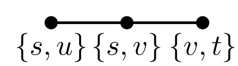
then $\mathcal{D}_{even}$ is given by



- $\mathcal{D}_{\bullet\bullet}$ is the diagram with vertex set $\{\{s,t\} \mid s,t \in S, m(s,t) = 2\}$. There is an edge between $\{s_1,t_1\}$ and $\{s_2,t_2\}$ in $\mathcal{D}_{\bullet\bullet}$ if $s_1 = s_2$ and $m(t_1,t_2)$ is odd. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



then $\mathcal{D}_{\bullet\bullet}$ is given by



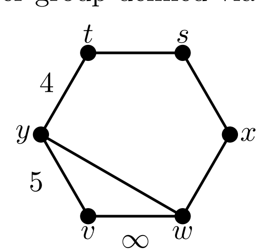
Theorem A. *Given a finite rank Coxeter group W, there is a natural isomorphism*

$$H_2(W;\mathbb{Z}) \cong H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2)$$

where in the first and final term of the right-hand-side the diagrams are considered as simplicial complexes consisting of 0-simplices (vertices of the diagram) and 1-simplices (edges of the diagram).

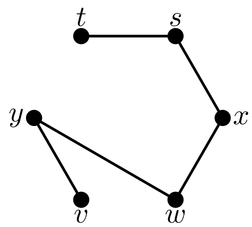
Computing the rank of the right hand side recovers Howlett's theorem [**32**].

Example. Let W be the Coxeter group defined via the following diagram



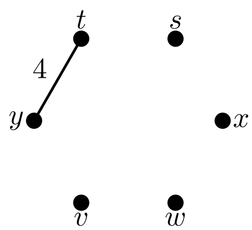
where we choose this example as it relates to an infinite Coxeter group which is not one of the classically studied Coxeter groups. Then the subdiagrams and consequent simplicial homologies representing the second integral homology of W are:

$\mathcal{D}_{odd}$:



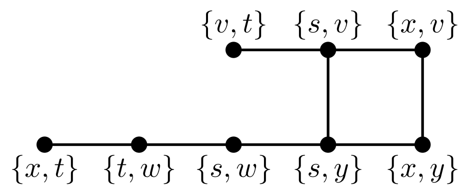
$H_1(\mathcal{D}_{odd}; \mathbb{Z}_2) = 0$

$\mathcal{D}_{even}$:



$\mathbb{Z}_2[E(\mathcal{D}_{even})] = \mathbb{Z}_2$

$\mathcal{D}_{\bullet\bullet}$:



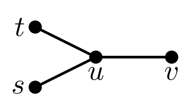
$H_0(\mathcal{D}_{\bullet\bullet}; \mathbb{Z}_2) = \mathbb{Z}_2$

and, hence, Theorem A yields

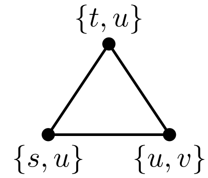
$$H_2(W; \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Our second theorem computes the third integral homology of a finitely generated Coxeter group. To state this theorem we introduce four new diagrams, $\mathcal{D}_{A_2}$, $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$, $\mathcal{D}_{A_3}$ and $\mathcal{D}^{\square}_{\bullet\bullet}$.

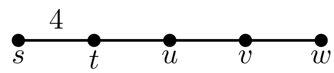
- $\mathcal{D}_{A_2}$ is the diagram with vertex set $\{\{s,t\} \mid s,t \in S,\ m(s,t) = 3\}$. There is an edge between $\{s_1, t_1\}$ and $\{s_2, t_2\}$ in $\mathcal{D}_{A_2}$ if $s_1 = s_2$ and $m(t_1, t_2) = 2$. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



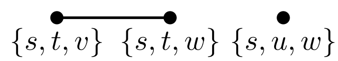
then $\mathcal{D}_{A_2}$ is given by



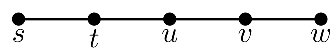
- $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is the diagram with vertex set $\{\{s,t,u\} \mid s,t,u \in S,\, m(s,t) = m(s,u) = 2 \text{ and } m(t,u) \text{ is even}\}$. There is an edge between $\{s_1,t_1,u_1\}$ and $\{s_2,t_2,u_2\}$ in $\mathcal{D}_{A_2}$ if $t_1 = t_2$, $u_1 = u_2$ and $m(s_1,s_2)$ is odd. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



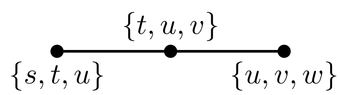
then $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is given by



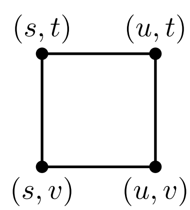
- $\mathcal{D}_{A_3}$ is the diagram with vertex set $\{\{s,t,u\} \mid s,t,u \in S,\, m(s,t) = m(t,u) = 3 \text{ and } m(s,u) = 2\}$. There is an edge between $\{s_1,t_1,u_1\}$ and $\{s_2,t_2,u_2\}$ in $\mathcal{D}_{A_3}$ if $t_1 = t_2$, $u_1 = u_2$ and $m(s_1,s_2) = 2$. For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



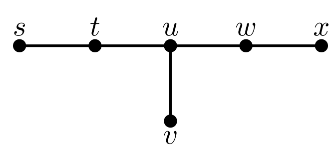
then $\mathcal{D}_{A_3}$ is given by



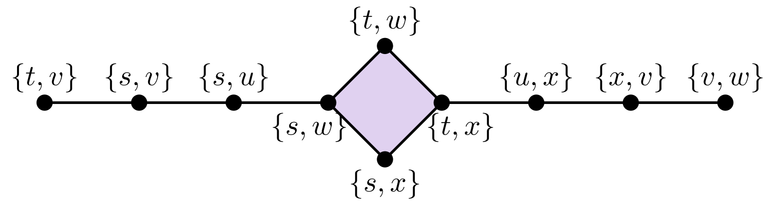
- $\mathcal{D}^{\square}_{\bullet\bullet}$ is the CW complex formed from the diagram $\mathcal{D}_{\bullet\bullet}$ by attaching a 2-cell to every square. Squares in $\mathcal{D}_{\bullet\bullet}$ have the form



For example when W is the Coxeter group with $\mathcal{D}_W$ the following diagram



then $\mathcal{D}^{\square}_{\bullet\bullet}$ is given by



Theorem B. *Given a finite rank Coxeter group W such that $\mathcal{D}_W$ does not have a subdiagram of the form $Y \sqcup A_1$, where Y is a loop in the Coxeter diagram $\mathcal{D}_{odd}$, there is an isomorphism*

$$\begin{aligned} H_3(W;\mathbb{Z}) \quad \cong \quad & H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus \Big(\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}\Big) \oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \\ & \oplus \Big(\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2\Big) \oplus \big(H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\big), \end{aligned}$$

where each diagram is as described above, and viewed as a simplicial complex. In this equation, $\bigcirc$ denotes a known non-trivial extension of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ fully described via an extension matrix X_W.

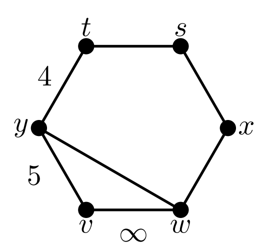
If W is such that $\mathcal{D}_W$ has a subdiagram of the form $Y \sqcup A_1$ where Y is a 1-cycle in the Coxeter diagram $\mathcal{D}_{odd}$, then there is an isomorphism modulo extensions

$$\begin{aligned} H_3(W;\mathbb{Z}) \quad \cong \quad & H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus \Big(\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}\Big) \oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \\ & \oplus \Big(\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2\Big) \oplus \big(H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\big) \\ & \oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2), \end{aligned}$$

where the unknown extensions involve the $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ summand.

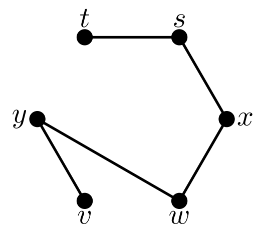
The diagrams appearing on the right hand side of the isomorphism are relatively simple to compute, as shown in the below example.

Example. Let W be, again, the Coxeter group defined via the following diagram



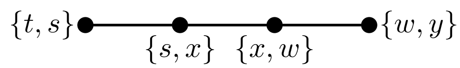
Then the subdiagrams and consequent simplicial homologies representing the third integral homology of W are:

$\mathcal{D}_{odd}$:



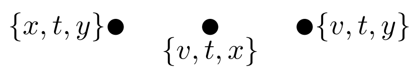
$H_0(\mathcal{D}_{odd}; \mathbb{Z}_2) = \mathbb{Z}_2$

$\mathcal{D}_{A_2}$:



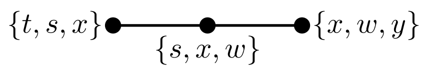
$H_0(\mathcal{D}_{A_2}; \mathbb{Z}_3) = \mathbb{Z}_3$

$\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$



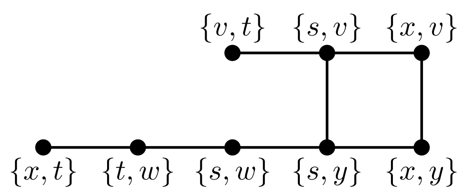
$H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\mathcal{D}_{A_3}$:



$H_0(\mathcal{D}_{A_3}; \mathbb{Z}_2) = \mathbb{Z}_2$

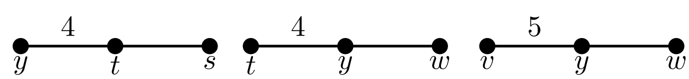
$\mathcal{D}_{\bullet\bullet}$:



$H_0(\mathcal{D}_{\bullet\bullet}; \mathbb{Z}_2) = \mathbb{Z}_2$

and Theorem B also requires us to count edges with label bigger than 3 but not infinity:

and subdiagrams of particular shapes, of which we have three:



We note that $\mathcal{D}_{odd}$ has no loop in it. Putting this all together, for the Coxeter group W related to $\mathcal{D}_W$ we have from Theorem B that the third integral homology is the sum of the right hand column, with a known non-trivial extension, plus a summand for each of the edges and the subdiagrams highlighted above

$$H_3(W; \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_5) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

These results arise from the computation of the *isotropy spectral sequence*, for a contractible CW-complex upon which the Coxeter group acts, called the *Davis complex*. The spectral sequence computations rely heavily on a free resolution for Coxeter groups, described by De Concini and Salvetti in *Cohomology of Coxeter groups and Artin groups* [**18**]. The computer algebra package PyCox is used to complete some of the longer calculations required, and an overview of this Python package can be found in *PyCox: Computing with (finite) Coxeter groups and Iwahori-Hecke algebras* by Geck [**26**].

We note here that in an unpublished paper *Cohomology of some Artin groups and monoids* by Ellis and Sköldberg [**23**], they remark on page 20 that the PhD thesis *Homology of Coxeter groups and related calculations* by J. Harris at NUI Galway contains a calculation of the third integral homology of a Coxeter group. This remark is also mirrored in Example 3 of *Polytopal resolutions for finite groups* by by Ellis, Harris and Sköldberg [**22**].

Results: Homological stability for Artin Monoids

The main influencing factor for selecting this topic of study, as well as the inspiration for much of the set up for the proof, was Hepworth's *Homological Stability for Families of Coxeter Groups* [**31**].

For every Coxeter group W there is a corresponding Artin group A_W with presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \,|\, \pi(\sigma_s, \sigma_t; m(s,t)) = \pi(\sigma_t, \sigma_s; m(s,t)),\, \forall s, t \in S \rangle.$$

Here we note that the Coxeter diagram $\mathcal{D}_W$ also contains all the information on the Artin group presentation. The *Artin monoid* A_W^+ of an Artin group A_W associated to a Coxeter group W is defined to be the monoid with the same presentation as A:

$$A_W^+ = \langle \sigma_s \text{ for } s \in S \,|\, \pi(\sigma_s, \sigma_t; m(s,t)) = \pi(\sigma_t, \sigma_s; m(s,t)),\, \forall s, t \in S \rangle^+.$$

A family of groups or monoids

$$G_1 \to G_2 \to \cdots \to G_n \to \cdots$$

is said to satisfy *homological stability* if the induced maps on homology

$$H_i(BG_n) \to H_i(BG_{n+1})$$

are isomorphisms for n sufficiently large compared to i.

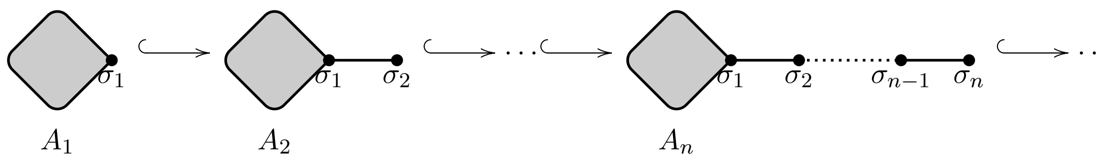
The topic of homological stability has been widely studied since the latter half of the twentieth century, with classical examples being homological stability for the sequence of: symmetric groups S_n by Nakaoka [**39**]; general linear groups GL_n by Maazen [**35**] and Van der Kallen [**45**]; and braid groups $\mathcal{B}_n$ by Arnol'd [**4**]. These classical examples are all proofs of homological stability for sequences of discrete groups, but the scope of homological stability results is much broader than this, and there are numerous examples of groups and spaces which satisfy homological stability and closely related phenomena. Recently, work of Basterra, Bobkova, Ponto, Tillmann and Yeakel, defines and studies homological stability for operads

[**6**], and work of Galatius and Randal-Williams [**25**] has focused on homological stability results for moduli spaces of manifolds.

In many cases where homological stability is known it is difficult to compute the homology of a group in the sequence in general. However there are techniques to compute the *stable homology* of the sequence and due to the homological stability result this gives us infinitely many new computations of the group homology. The question of the stable homology is not addressed in this thesis.

The theory of homological stability has been enclosed in a generalised framework during the past few years. Recent work by Randal-Williams and Wahl [**42**] presents a categorical framework for homological stability results for discrete groups, and work of Krannich [**34**] generalises this to a framework in the context of E_2-algebras. However both of these frameworks still require a proof of high connectivity, arguably the most difficult and non-standard part of a homological stability proof, to be inserted in order to yield results.

Our result concerns a sequence of Artin monoids with the braid monoid as a sub-monoid. The maps are given by inclusions corresponding to increasing the number of generators of the braid sub-monoid. In this case the sequence of Coxeter diagrams relating to the corresponding Artin groups is as follows



Theorem C. *The sequence of Artin monoids*

$$A_1^+ \hookrightarrow A_2^+ \hookrightarrow \cdots \hookrightarrow A_n^+ \hookrightarrow \cdots$$

satisfies homological stability. That is, the induced map on homology

$$H_*(BA_{n-1}^+) \xrightarrow{s_*} H_*(BA_n^+)$$

is an isomorphism when $* < \frac{n}{2}$ *and a surjection when* $* = \frac{n}{2}$. *Here homology is taken with arbitrary constant coefficients.*

As far as we are aware, this is the only homological stability proof for a sequence of monoids that are not groups, though often homological stability results for groups will imply results on the homology of associated monoids. In particular when the *Ore condition* holds there is a homotopy equivalence between the classifying spaces of the group and the monoid. In our case, this equivalence is true if and only if the $K(\pi, 1)$ conjecture holds and therefore we deduce a homological stability result in an unconventional direction: from monoids to groups.

Corollary D. *Suppose the* $K(\pi, 1)$ *conjecture holds for the sequence of Artin groups*

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots$$

then the sequence satisfies homological stability, with the same range as in Theorem C.

Homological stability was demonstrated for the finite type families of Artin groups of the form we study, via a computation of their full cohomology by Arnol'd, written in the Bourbaki paper *Sur les groupes des tresses* by Brieskorn [**10**].

The key step in the proof of the theorem is to show that a certain family of semi-simplicial spaces on which the monoids A_n^+ act is highly connected. To define this family of spaces and prove the related connectivity requires simplicial set theory, following the recent and very useful text *Semi-simplicial spaces* by Ebert and Randal-Williams [**21**]. The proof of high connectivity follows a *union of chambers argument*, as in many proofs of homological stability. This argument was particularly influenced by a high connectivity argument in Paris's notes on the $K(\pi, 1)$ *conjecture for Artin groups* [**40**]. This argument comprises the most technical part of the proof and utilises monoid theory, in particular following theory for Artin monoids from Brieskorn and Saito's *Artin Gruppen und Coxeter Gruppen* [**11**].

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CHAPTER 1

Background: Coxeter groups

This chapter follows *The Geometry and Topology of Coxeter Groups* by Davis [**17**].

1.1. Definition and examples

DEFINITION 1.1.1. A *Coxeter matrix* on a finite set of generators S, is a symmetric matrix M indexed by elements of S, i.e. with integer entries $m(s,t)$ for $\{s,t\}$ in $S \times S$. This matrix must satisfy

- $m(s,s) = 1$ for all s in S
- $m(s,t) = m(t,s)$
- $m(s,t)$ must be either greater than 1, or ∞, when $s \neq t$.

DEFINITION 1.1.2. A Coxeter matrix M has an associated *Coxeter group*, W, with presentation

$$W = \langle S \,|\, \forall s,t \in S, (st)^{m(s,t)} = e \rangle.$$

We call (W,S) a *Coxeter system*, and we call $|S|$ the *rank* of W. We adopt the convention that $(W,\emptyset)$ is the trivial group.

REMARK 1.1.3. Note that the condition $m(s,s) = 1$ on the Coxeter matrix implies that the generators of the group are involutions i.e., $s^2 = e$ for all s in S.

DEFINITION 1.1.4. Define the *length function* on a Coxeter system (W,S)

$$\ell : W \to \mathbb{N}$$

to be the function which maps w in W to the minimum word length required to express w in terms of the generators. That is, we set $\ell(e) = 0$, and if $w \neq e$ is in W then there exists a $k \geq 1$ such that $w = s_1 \dots s_k$ for s_i in S. We choose minimal k satisfying this property and set $\ell(w) = k$.

EXAMPLE 1.1.5. If $m(s,t) \geq 3$ then $\ell(sts) = 3$ whereas if $m(s,t) = 2$ then

$$sts = s(ts) = s(st) = (s^2)t = t$$

and so $\ell(sts) = \ell(t) = 1$.

Definition 1.1.6. Define $\pi(a,b;k)$ to be a word of length k, given by the alternating product of a and b i.e.

$$\pi(a,b;k) = \overbrace{abab\dots}^{\text{length k}}$$

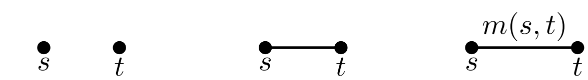
Remark 1.1.7. The relations $(st)^{m(s,t)} = e$ can be rewritten by left multiplying by s and t in turn and using the fact that the generators are involutions to get

$$\pi(s,t;m(s,t)) = \pi(t,s;m(s,t))$$

when $m(s,t) \neq \infty$. For example when $m(s,t) = 3$, the relation $(st)^3 = e$ can be rewritten as $sts = tst$. Therefore the presentation of a Coxeter group W can also be given as

$$W = \left\langle S \;\middle|\; \begin{array}{cc} (s)^2 = e & \forall s \in S \\ \pi(s,t;m(s,t)) = \pi(t,s;m(s,t)) & \forall s,t \in S \end{array} \right\rangle.$$

Definition 1.1.8. Given a Coxeter matrix corresponding to a Coxeter system (W,S), there is an associated graph called the *Coxeter diagram*, denoted $\mathcal{D}_W$. It is the graph with vertices indexed by the elements of the generating set S. Edges are drawn between the vertices corresponding to s and t in S when $m(s,t) \geq 3$ and labelled with $m(s,t)$ when $m(s,t) \geq 4$, as shown below:

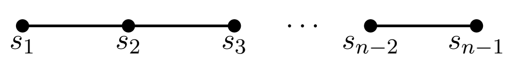


$m(s,t) = 2$ $\qquad$ $m(s,t) = 3$ $\qquad$ $m(s,t) \geq 4$

When the diagram $\mathcal{D}_W$ is connected, W is called an *irreducible* Coxeter group.

Example 1.1.9. The Coxeter group with one generator $W = \langle s | s^2 = e \rangle$ is the cyclic group of order 2. We call this the Coxeter group *of type* A_1. Its diagram $\mathcal{D}_W$ is given by

Example 1.1.10. The symmetric group S_n is an example of a Coxeter group: it is isomorphic to the Coxeter group of *type* A_{n-1}, which has the following diagram



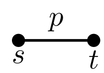
We consider the isomorphism which sends a generator s_i to the transposition $(i,\ i+1)$. If two transpositions are disjoint they commute, whereas transpositions

$$s_i s_{i+1} = (i,\ i+1)(i+1,\ i+2) = (i,\ i+1,\ i+2)$$

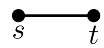
form a 3-cycle and therefore $s_i s_{i+1}$ has order 3 for all $1 \leq i \leq n-2$. This corresponds to the relations given by the Coxeter diagram of type A_{n-1}, for the symmetric group presentation:

$$S_n = \langle \{s_1, \dots s_{n-1}\} \,|\, s_i^2 = e \ \forall i, (s_i s_j)^2 = e \ \forall |i-j| \geq 2, (s_i s_{i+1})^3 = e \ \forall 1 \leq i \leq (n-2) \rangle.$$

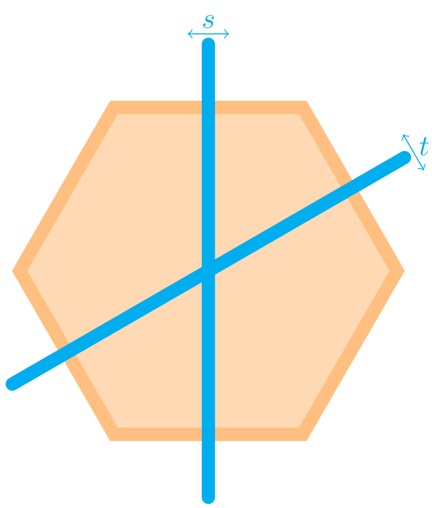
EXAMPLE 1.1.11. The dihedral group D_{2p}, of order $2p$, is an example of a Coxeter group: it is isomorphic to the Coxeter group of *type* $I_2(p)$, which has the following diagram:



and here we note that if p is 2 then the edge is not included in the diagram. The group D_{2p} can be viewed as the group of symmetries of a $2p$-gon, and to present it as a Coxeter group we exhibit a set of generating reflections. For instance the Coxeter group of type $I_2(3)$ has the following diagram



and correspondingly the dihedral group D_6 can be generated by reflections on the hexagon as depicted in the diagram below:



where we note that the reflections s and t both have order 2, and composing the reflections corresponds to rotation by $2\pi/3$, so $(st)^3$ is the identity. This agrees with the labels (or lack thereof) in the Coxeter diagram, and corresponds to the following presentation of D_6:

$$D_6 = \langle \{s, t\} \mid s^2 = t^2 = e,\, (st)^3 = e \rangle.$$

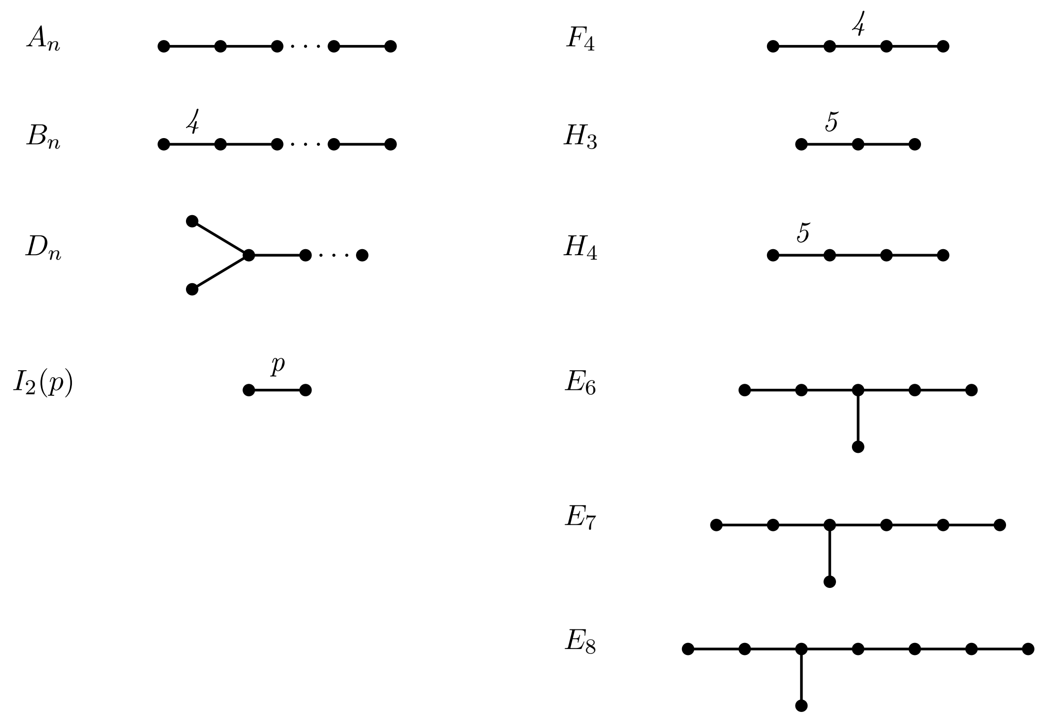
The examples we have considered have been those of finite Coxeter groups though of course, Coxeter groups are usually infinite (for instance any Coxeter group with an ∞ in the corresponding Coxeter matrix is infinite). There is also a notion of Coxeter groups with an infinite number of generators, but we do not approach this in this thesis. Coxeter completely classified the irreducible finite Coxeter groups in 1935 [**15**]. There are four infinite families of finite Coxeter groups, and six exceptional finite Coxeter groups.

THEOREM 1.1.12 (Classification of finite Coxeter groups, Coxeter [**15**]).

A Coxeter group is finite $\iff$ *it is the (direct) product of finitely many finite irreducible Coxeter group*

The following is a complete list of the diagrams corresponding to finite irreducible Coxeter groups, and therefore completely classifies finite Coxeter groups.

Infinite families *Exceptional groups*



Definition 1.1.13. We say that a finite irreducible Coxeter group W is *of type* $\mathcal{D}$ if its corresponding diagram is given by $\mathcal{D}$, and we denote this Coxeter group $W(\mathcal{D})$.

As we have seen in Examples 1.1.10 and 1.1.11, the Coxeter group of type A_n, or $W(A_n)$, corresponds to the symmetric group S_{n+1} and the Coxeter group of type $I_2(p)$, or $W(I_2(p))$, corresponds to the dihedral group D_{2p}. Similarly, the Coxeter group of type B_n, or $W(B_n)$, corresponds to the signed permutation group $\mathbb{Z}_2 \wr S_n$ (the A_{n-1} subdiagram present inside the diagram for B_n corresponds to the S_n subgroup of $\mathbb{Z}_2 \wr S_n$). The Coxeter group of type D_n, or $W(D_n)$, corresponds to an index two subgroup of type B_n, such that the signs in each permutation multiply to $+1$ (sign changes are even).

1.2. Products and subgroups

Consider two Coxeter systems (U, S_U) and (V, S_V). We will denote $\mathcal{D}_U \sqcup \mathcal{D}_V$ by the diagram created by placing the two corresponding diagrams $\mathcal{D}_U$ and $\mathcal{D}_V$ beside each other, disjointly.

LEMMA 1.2.1. *With notation as above, the diagram $\mathcal{D}_U \sqcup \mathcal{D}_V$ corresponds to taking a product of Coxeter groups $U \times V$, and defines another Coxeter group $W \cong U \times V$, which has diagram $\mathcal{D}_W = \mathcal{D}_U \sqcup D_V$ and generating set $S_W := S_U \cup S_V$. The Coxeter relations are given by those for (U, S_U) and (V, S_V), and letting $m(s_u, s_v) = 2$ for all s_u in S_U and s_v in S_V.*

PROOF. The generating set and relations for (W, S_W) can be read off the Coxeter diagram $\mathcal{D}_W = \mathcal{D}_U \sqcup \mathcal{D}_V$. In particular, since there are no edges between the subdiagram $\mathcal{D}_U$ and the subdiagram $\mathcal{D}_V$, $m(s_u, s_v) = 2$ for all s_u in S_U and s_v in S_V. Since the generators from S_U and S_V commute pairwise, any word w in W can be written as $w = uv$ for u in U and v in V. Then the group W is isomorphic to the product group $U \times V$ via the map

$$\begin{array}{rcl} W & \cong & U \times V \\ w = uv & \longleftrightarrow & (u, v). \end{array}$$

□

EXAMPLE 1.2.2. The finite Coxeter group of type $I_2(2)$ is an example of a product of Coxeter groups. Its diagram has the form

$$\underset{s}{\bullet} \qquad \underset{t}{\bullet}$$

and so it is in fact isomorphic to the product of the Coxeter group of type A_1 with itself: the group $W(A_1) \times W(A_1)$. The product group has two generators, both with order 2, that commute, and is therefore isomorphic to the product of cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_2$.

DEFINITION 1.2.3. We say that an inclusion of Coxeter diagrams $\mathcal{D}_U \overset{\iota}{\hookrightarrow} \mathcal{D}_W$ is *full* if for every two vertices s and t in $\mathcal{D}_U$, $m(s, t)$ is the same in $\mathcal{D}_W$ as it is in $\mathcal{D}_U$. In other words, if two generators are in S_U then they are also in S_W (via the inclusion map) and we insist that the edge between them is the same in $\mathcal{D}_U$ as it appears in $\mathcal{D}_W$. In this setting we call $\mathcal{D}_U$ a *full subdiagram* of $\mathcal{D}_W$.

DEFINITION 1.2.4. Let (W, S) be a Coxeter system. For each $T \subseteq S$ denote by W_T the subgroup of W generated by T. Denote the diagram corresponding to this subgroup by $\mathcal{D}_{W_T}$. We call subgroups that arise in this way *parabolic subgroups*.

PROPOSITION 1.2.5 (see Davis [**17**, 4.1.6.(i)]). *For W_T a parabolic subgroup, (W_T, T) is a Coxeter system in its own right, and defines a full inclusion $\mathcal{D}_{W_T} \hookrightarrow \mathcal{D}_W$. Similarly, a full inclusion corresponding to a parabolic subgroup.*

Throughout this writing, many of the results and theory are inspired by or correspond to the theory of cosets in Coxeter groups. The next result concerns cosets of parabolic subgroups. Let (W, S) be a Coxeter system, and T, T' be subsets of S.

Lemma 1.2.6 (see Davis [**17**, 4.3.1]). *There is a unique element w of minimum length in the double coset $W_T w W_{T'}$. More precisely, any element in this double coset can be written as awa' where a is in W_T, a' is in $W_{T'}$ and $\ell(awa') = \ell(a) + \ell(w) + \ell(a')$.*

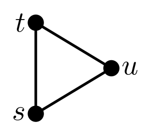
Definition 1.2.7 (see Davis [**17**, 4.3.2]). We say an element w in W is (T, T')-reduced if w is the shortest element in $W_T w W_{T'}$.

Remark 1.2.8. Given the parabolic subgroup W_T in W, w in W is $(T, \emptyset)$-reduced if $\ell(tw) = \ell(t) + \ell(w) = 1 + \ell(w)$ for all t in T. Note that this means that the word w cannot be rearranged to start with the letter t. Likewise we say w in W is $(\emptyset, T)$-reduced if $\ell(wt) = \ell(w) + 1$ for all t in T. Similarly this means that the word w cannot be rearranged to end with the letter t.

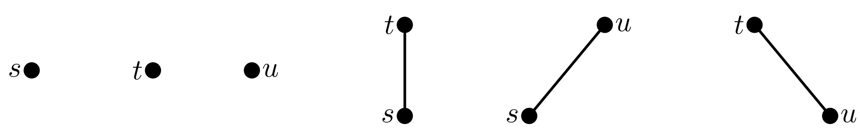
Definition 1.2.9. A finite parabolic subgroup is called a *spherical subgroup*.

Since the diagrams of parabolic subgroups appear as full subdiagrams of the Coxeter diagram, for a Coxeter system (W, S) we can identify its spherical subgroups by identifying occurrences of the irreducible diagrams from Theorem 1.1.12 in $\mathcal{D}_W$, and disjoint unions of such diagrams.

Example 1.2.10. Consider the Coxeter group W corresponding to the following diagram



Then W is infinite: one way to view this is by considering W as the group of symmetries of the Euclidean plane tiled by equilateral triangles, with generators s, t and u corresponding to reflections in the three edges of a chosen 'fundamental' triangle. Then for any other triangle in this tiling there is a word in W mapping the fundamental triangle to the chosen triangle, and so one can observe that the group is infinite. The spherical subgroups of W are given by the following subdiagrams (of type A_1 and A_2), as well as the trivial group $W_\emptyset$.



Definition 1.2.11. We denote by $\mathcal{S}$ the set of all subsets of S which generate spherical subgroups of W, i.e.

$$\mathcal{S} = \{T \subset S \mid W_T \text{ is finite}\}.$$

We will sometimes refer to an element T of $\mathcal{S}$ as a *spherical subset.*

Remark 1.2.12. Let s, t in S. We note that every one-generator subgroup $W_{\{s\}}$ for s in S satisfies that $W_{\{s\}}$ is of type A_1, and so is finite. For the remainder of this thesis we write

W_s for $W_{\{s\}}$. Furthermore when $m(s,t) \neq \infty$, $W_{\{s,t\}}$ is of type $I_2(m(s,t))$, which is a finite subgroup, so every edge not labelled by ∞ in $\mathcal{D}_W$ represents a finite group. Finally we note that, since we adopted the convention that the group with no generators and no relations is the (finite) trivial group, $\emptyset$ is always present in $\mathcal{S}$.

LEMMA 1.2.13 (see Davis [**17**, 4.6.1]). *If W is a finite Coxeter group generated by S, there is a unique element w_0 of longest length in W, satisfying $\ell(sw_0) < \ell(w_0)$ for all s in S.*

It follows that every spherical subgroup W_T of a Coxeter group W has a longest element.

1.3. The Davis complex

Recall that subsets of S generate subgroups of W and these are known as *parabolic subgroups*, denoted W_T, for T a subset of S. If a parabolic subgroup is finite we call it a *spherical subgroup* and we denote the set of all subsets of S which generate spherical subgroups of W by $\mathcal{S}$.

DEFINITION 1.3.1. A coset of a spherical subgroup is called a *spherical coset*. For a Coxeter system (W,S) and a subgroup W_T we denote the set of cosets as follows:

$$W/W_T = \{wW_T \mid w \in W\}.$$

The set of all spherical cosets is denoted $W\mathcal{S}$:

$$W\mathcal{S} = \bigcup_{T\in\mathcal{S}} W/W_T.$$

$W\mathcal{S}$ is partially ordered by inclusion and so can be considered as a poset. The group W acts on the poset $W\mathcal{S}$ by left multiplication and the quotient poset is $\mathcal{S}$.

LEMMA 1.3.2 (see Davis [**17**, 4.1.6.(iii)]). *Given T and U in $\mathcal{S}$ and w and v in W, the cosets wW_U and vW_T satisfy $wW_U \subseteq vW_T$ if and only if $w^{-1}v \in W_T$ and $U \subseteq T$.*

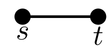
DEFINITION 1.3.3 (see Davis [**17**, 7.2]). We can associate to any poset $\mathcal{P}$, its *geometric realisation*. This is given by the geometric realisation of the abstract simplicial complex $Flag(\mathcal{P})$ which consists of all finite chains in $\mathcal{P}$. The reader is directed to Appendix A of Davis for more details.

DEFINITION 1.3.4 (see Davis [**17**, 7.2]). One can associate to a Coxeter group a CW complex called the *Davis Complex*. This is denoted Σ_W and is the geometric realisation of the poset $W\mathcal{S}$. That is every spherical coset wW_T is realised as a vertex or 0-simplex, and for every ordered chain of $(p+1)$ spherical cosets, with $p \geq 0$ there is a p-simplex in the Davis Complex:

$$w_0W_{T_0} \subset w_1W_{T_1} \subset w_2W_{T_2} \subset \cdots \subset w_pW_{T_p}$$

where here w_i is in W and T_i is in $\mathcal{S}$ for all $0 \leq i \leq p$.

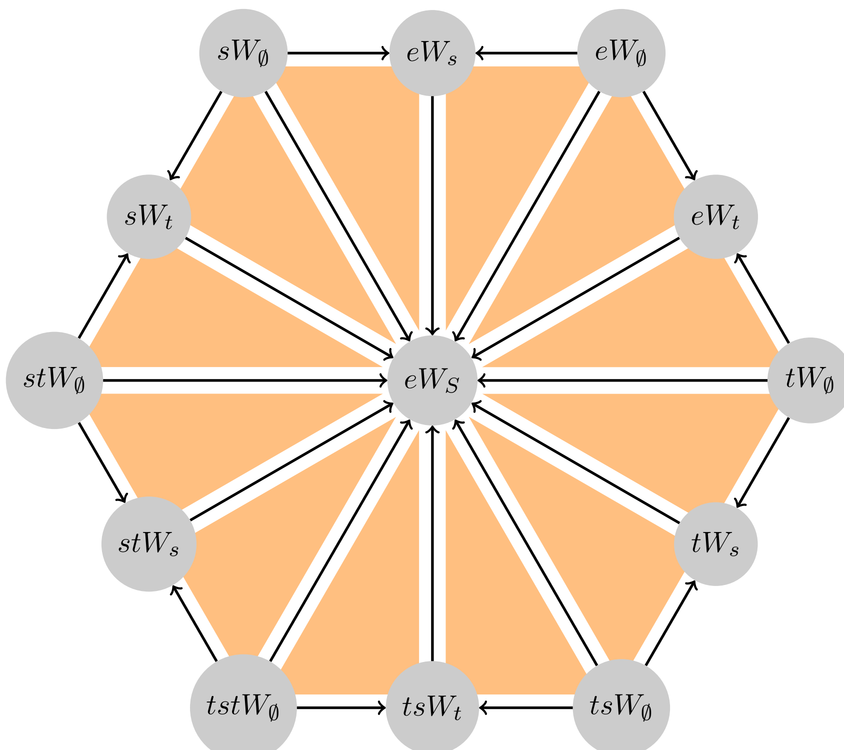
EXAMPLE 1.3.5. We work through the construction of the Davis complex for the Coxeter group $W = W(I_2(3))$ which we recall from Example 1.1.11 to be the dihedral group D_6. Then $\mathcal{D}_W$ is given by



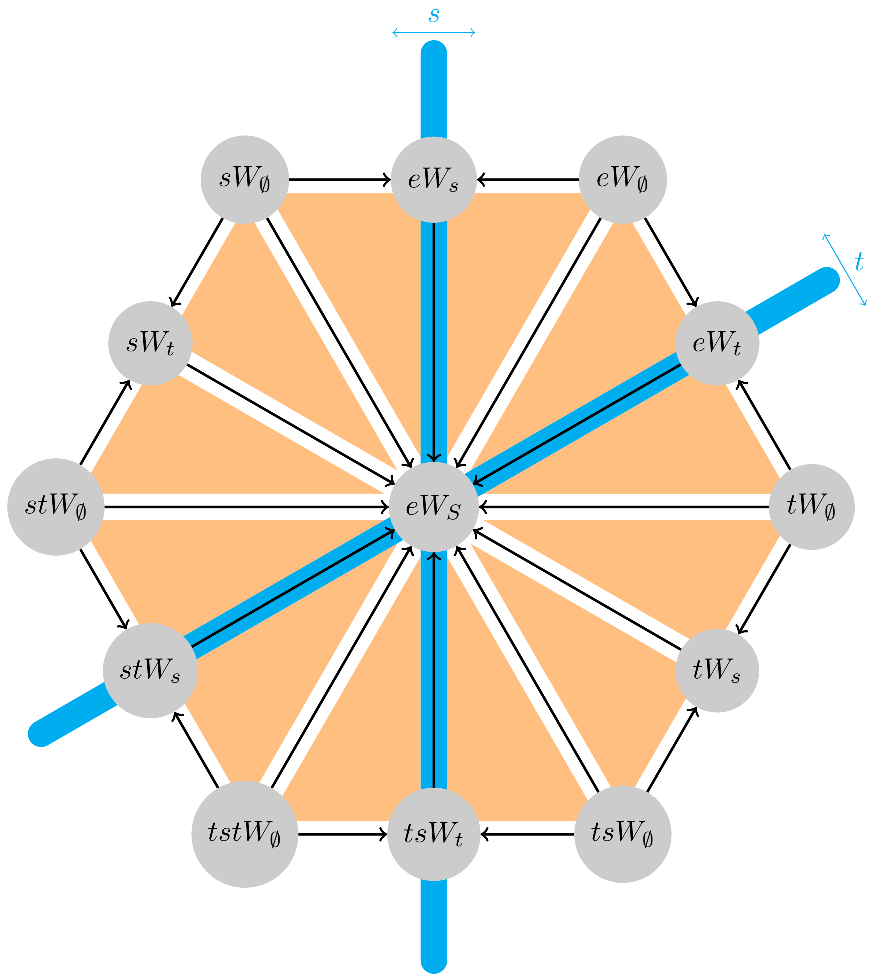
and so spherical subgroups are given by the subdiagrams that correspond to finite subgroups, that is

$$\mathcal{S} = \{\emptyset, s, t, S\}.$$

Considering the spherical cosets and inclusions, we have, for example, $eW_\emptyset \subset eW_s \subset eW_S$ and so a 2-simplex is formed. Considering all such inclusions and constructing the Davis complex gives the following:



where the circles symbolise vertices, the arrows symbolise inclusions and 1-simplices and the orange triangles symbolise 2-simplices. The Coxeter group $W = W(I_2(3))$ acts on the Davis complex by left multiplication of the cosets and the action of the two generators s and t on the complex is shown below in blue:



Definition 1.3.6 (see Davis [**17**, A.1.1]). A *convex polytope* in an affine space $\mathbb{A}$ is the convex hull of a finite subset of $\mathbb{A}$. Its dimension is given by the dimension of the subspace of $\mathbb{A}$ which it spans. Equivalently, a convex polytope may be defined as the compact intersection of a finite set of half spaces in $\mathbb{A}$.

Remark 1.3.7. A 0-dimensional convex polytope is a point, a 1-dimensional convex polytope is a line segment, and a 2-dimensional convex polytope is a polygon. In general, a k-dimensional convex polytope is homeomorphic to a k-disk.

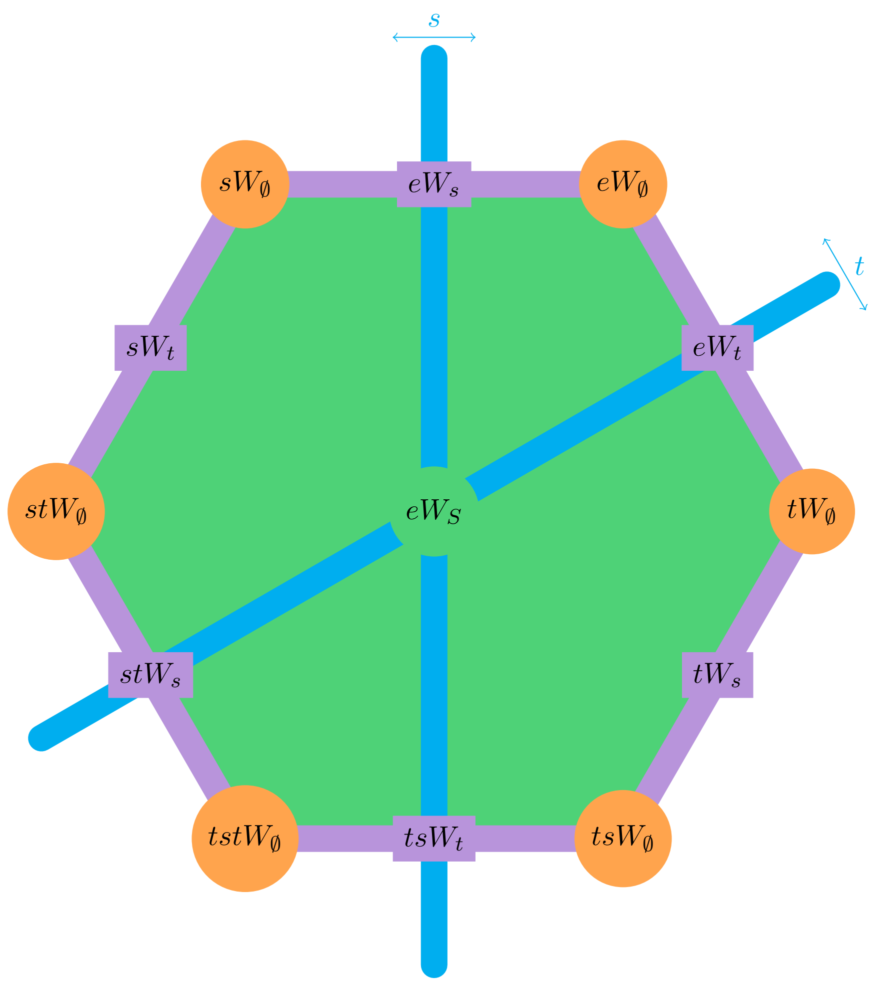
Definition 1.3.8. For every finite Coxeter group W with generating set S, one can define a *canonical representation* of the Coxeter group W on $\mathbb{R}^n$, where $n = |S|$ (see Davis section 6.12 for details). Given this representation, we define the *Coxeter polytope*, or *Coxeter cell* of W to be the convex hull of the orbit of a generic point x in $\mathbb{R}^n$ under the W-action. This polytope has dimension $n = |S|$, and we denote it C_W. A detailed definition can be found in Davis section 7.3 [**17**].

PROPOSITION 1.3.9. *If W is a finite Coxeter group then Σ_W, the geometric realisation of $W\mathcal{S}$, is isomorphic to the barycentric subdivision of the Coxeter cell C_W.*

PROOF. The proof follows from Davis Lemma 7.3.3 [**17**]. □

DEFINITION 1.3.10. A coarser cell structure can be given to Σ_W by considering only those spherical cosets which are present as subsets of a particular coset wW_T. This is denoted $W\mathcal{S}_{\leq wW_T}$, and the realisation of this poset is a subcomplex of the realisation of $W\mathcal{S}$, i.e. a subcomplex of Σ_W. In fact $W\mathcal{S}_{\leq wW_T} \cong W_T\mathcal{S}_T$ where $\mathcal{S}_T$ denotes the set of spherical subsets of T. Since W_T is finite, the realisation of $W_T\mathcal{S}_T$, is isomorphic to the barycentric subdivision of its Coxeter cell C_{W_T}. Therefore the realisation is homeomorphic to a disk, i.e. $|W_T\mathcal{S}_T| \cong \mathbb{D}^{|T|}$. The cell structure on Σ_W is therefore given by associating to the subcomplex $W\mathcal{S}_{\leq wW_T}$ its corresponding Coxeter cell: a p-cell where $p = |T|$. The 0-cells are given by cosets of the form $W\mathcal{S}_{\leq wW_\emptyset}$, i.e. the set $\{wW_\emptyset | w \in W\}$, and therefore associated to elements of W (recall $W_\emptyset = \{e\}$). By Lemma 1.3.2 a set of vertices X will define a p-cell precisely when $X = \{v \in W | v \in wW_T\}$ for $T \in \mathcal{S}$ and $|T| = p$. There is an action of W on the cells of Σ_W given by left multiplication, and this permutes the cells.

EXAMPLE 1.3.11. We consider the above cell structure for our running example of $W = W(I_2(3))$, noting the action of the generators of W in blue. There are six 0-cells, six 1-cells and one 2-cell, corresponding to spherical cosets with generating sets having 0, 1 and 2 elements respectively.



Alongside the formulation and cell structure, we use the following results concerning the Davis complex in Chapter 2.

Proposition 1.3.12 (Davis [**17**, 8.2.13]). *For any Coxeter group W, Σ_W is contractible.*

Lemma 1.3.13 (Davis [**17**, 7.4.4]). *If W and S decompose as $W = U \times V$ and $S = S_U \cup S_V$ then $\mathcal{S} = \mathcal{S}_U \times \mathcal{S}_W$ and $\Sigma_W = \Sigma_U \times \Sigma_V$.*

CHAPTER 2

Results: Low dimensional homology of Coxeter groups

In this chapter we prove two theorems which calculate the second and third integral homology of a finite rank Coxeter group. These results arise from the computation of the *isotropy spectral sequence*, for a contractible CW-complex upon which the Coxeter group acts, called the *Davis complex*. For the degree three result, the spectral sequence computations rely heavily on a free resolution for Coxeter groups, described by De Concini and Salvetti in *Cohomology of Coxeter groups and Artin groups* [**18**].

2.1. Discussion of results

Given a Coxeter system (W, S), let the corresponding Coxeter diagram be denoted $\mathcal{D}_W$. Let us first consider $H_1(W; \mathbb{Z}) = W_{\text{abelian}}$, the abelianisation of W.

DEFINITION 2.1.1 (see Brown [**12**, III.1]). Let G be a group and F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. For a G-module M we define the *group homology of G with coefficients in M* to be

$$H_*(G; M) = H_*(F \otimes_G M).$$

LEMMA 2.1.2. *Let (W, S) be a Coxeter system. Let $s \sim s'$ if there is a sequence $s = s_0, s_1, \ldots, s_n = s'$ of elements of S such that $m(s_i, s_{i+1})$ is an odd integer. Then $\sim$ defines an equivalence relation on S and it follows that s and s' represent the same element of the abelianisation of W if and only if $s \sim s'$.*

PROOF. From Lemma 3.3.3 in Davis [**17**], $s \sim s'$ if and only if s and s' are conjugate. Since conjugate generators must be sent to the same element of the abelianisation the proof follows. □

COROLLARY 2.1.3. *As a consequence of Lemma 2.1.2, $H_1(W; \mathbb{Z})$ can be described by deleting even or infinite edges from the Coxeter diagram and counting the connected components of the remaining diagram. If there are d components then it follows that*

$$H_1(W; \mathbb{Z}) = W_{abelian} = \mathbb{Z}_2^d.$$

In [**32**], Howlett considers the Schur multiplier - which in this case is isomorphic to the second homology group $H_2(W; \mathbb{Z})$ - of finite rank Coxeter groups. We describe the result below.

DEFINITION 2.1.4. Let $S_{\bullet\bullet} = \{\{s,t\}|m(s,t)=2\}$ be the set consisting of unordered pairs of commuting generators. Let $\{s,t\} \approx \{s,t'\}$ if both pairs belong to $S_{\bullet\bullet}$, and $m(t,t')$ is odd. Let $\sim$ be the equivalence relation on $S_{\bullet\bullet}$ generated by $\approx$.

Let $\mathcal{D}_{\bullet\bullet}$ be the graph with vertex set indexed by $S_{\bullet\bullet}$ and an edge between the two vertices corresponding to $\{s,t\}$ and $\{s,t'\}$ if $\{s,t\} \approx \{s,t'\}$. Then the equivalence classes of $\sim$ are given precisely by the connected components of $\mathcal{D}_{\bullet\bullet}$.

Let $\mathcal{D}_{odd}$ be the diagram obtained from $\mathcal{D}_W$ by deleting all edges with an even label, or with an ∞ label, and $\mathcal{D}_{even}$ similarly (here we also delete the unlabelled edges as they correspond to $m(s,t)=3$). Let $E(\mathcal{D}_W)$ and $V(\mathcal{D}_W)$ be the set of edges and set of vertices of $\mathcal{D}_W$ respectively. Let

- $n_1(W)$ be the number of vertices of $\mathcal{D}_W$
- $n_2(W)$ be the number of edges of $\mathcal{D}_W$ carrying a finite weight
- $n_3(W)$ be the number of equivalence classes of $\sim$ on $S_{\bullet\bullet}$
- $n_4(W)$ be the number of connected components of $\mathcal{D}_{odd}$.

THEOREM 2.1.5 (Howlett [**32**]). *The Schur multiplier of W is an elementary abelian 2-group with rank*

$$n_2(W) + n_3(W) + n_4(W) - n_1(W).$$

The first theorem we prove in this text is a refinement of Howlett's theorem, based on the isotropy spectral sequence for the Davis complex, including a naturality statement.

THEOREM 2.1.6. *Given a finite rank Coxeter group W with diagram $\mathcal{D}_W$, there is a natural isomorphism*

$$H_2(W;\mathbb{Z}) = H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2)$$

where in the first and final term of the right-hand-side the diagrams are considered as simplicial complexes consisting of 0-simplices (vertices of the diagram) and 1-simplices (edges of the diagram).

REMARK 2.1.7. The naturality statement comes from the fact that, given a full inclusion of Coxeter groups $U \hookrightarrow W$, there is a full inclusion of Coxeter diagrams $\mathcal{D}_U \hookrightarrow \mathcal{D}_W$ with respect to which the assignments $\mathcal{D} \mapsto \mathcal{D}_{odd}$, $\mathcal{D} \mapsto \mathcal{D}_{even}$ and $\mathcal{D} \mapsto \mathcal{D}_{\bullet\bullet}$ are natural. That is, a full inclusion $\mathcal{D}_U \hookrightarrow \mathcal{D}_W$ induces a full inclusion of the diagrams $\mathcal{D}_{odd}$, $\mathcal{D}_{even}$ and $\mathcal{D}_{\bullet\bullet}$. The naturality of simplicial homology $H_*(-;\mathbb{Z}_2)$ with respect to sub-complexes of simplicial complexes therefore induces a component wise natural map on the right hand side of the isomorphism.

PROPOSITION 2.1.8. *This theorem recovers Howlett's theorem.*

PROOF. We compute the rank of each of the summand on the right hand side of Theorem 2.1.6.

- $\mathrm{rank}(H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) = n_3(W)$ by Definition 2.1.4.

- $\operatorname{rank}(\mathbb{Z}_2[E(\mathcal{D}_{even})]) = |E(\mathcal{D}_{even})|$.
- $\operatorname{rank}(H_1(\mathcal{D}_{odd};\mathbb{Z}_2) = \operatorname{rank}(\ker(d_1)/\operatorname{im}(d_2)) = \operatorname{null}(d_1) - \operatorname{rank}(d_2)$ where d_1 and d_2 are the simplicial boundary maps:

$$C_2(\mathcal{D}_{odd}) \xrightarrow{d_2} C_1(\mathcal{D}_{odd}) \xrightarrow{d_1} C_0(\mathcal{D}_{odd})$$

$$0 \xrightarrow{d_2} \mathbb{Z}_2[E(\mathcal{D}_{odd})] \xrightarrow{d_1} \mathbb{Z}_2[V(\mathcal{D}_{odd})].$$

  It follows that
  - $\operatorname{rank}(d_1)$ grows by 1 for each vertex connected to an edge in $\mathcal{D}_{odd}$, subject to the relation that the vertices of a component of $\mathcal{D}_{odd}$ are identified (this decreases the dimension of the image by one for each non-trivial component of $\mathcal{D}_{odd}$). A vertex which is not connected to an edge in $\mathcal{D}_{odd}$ has its own component in $\mathcal{D}_{odd}$. Therefore $\operatorname{rank}(d_1) = n_1(W) - n_4(W)$.
  - $\operatorname{null}(d_1) + \operatorname{rank}(d_1) = \dim(C_1(\mathcal{D}_{odd})) = |E(\mathcal{D}_{odd})|$ so $\operatorname{null}(d_1) = |E(\mathcal{D}_{odd})| - \operatorname{rank}(d_1) = |E(\mathcal{D}_{odd})| - n_1(W) + n_4(W)$
  - $\operatorname{rank}(d_2) = 0$.
  - $\operatorname{null}(d_1) - \operatorname{rank}(d_2) = |E(\mathcal{D}_{odd})| - n_1(W) + n_4(W)$.

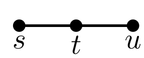
Therefore the rank on the right hand side of Theorem 2.1.6 is given by

$$\begin{aligned} & \operatorname{rank}(H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2)) \\ = \;& \operatorname{rank}(H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) + \operatorname{rank}(\mathbb{Z}_2[E(\mathcal{D}_{even})]) + \operatorname{rank}(H_1(\mathcal{D}_{odd};\mathbb{Z}_2))) \\ = \;& n_3(W) + |E(\mathcal{D}_{even})| + |E(\mathcal{D}_{odd})| - n_1(W) + n_4(W) \\ = \;& n_3(W) + (|E(\mathcal{D}_{even})| + |E(\mathcal{D}_{odd})|) - n_1(W) + n_4(W) \\ = \;& n_3(W) + n_2(W) - n_1(W) + n_4(W) \end{aligned}$$

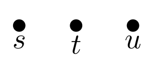
as required. $\square$

Example 2.1.9. An example of Theorem 2.1.6 for an infinite Coxeter group can be found in the introduction to this thesis.

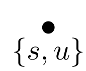
Example 2.1.10. When the Coxeter group W is the finite group of type A_3 we have that $\mathcal{D}_W$ is



and so $\mathcal{D}_{odd}$ is $\mathcal{D}_W$, $\mathcal{D}_{even}$ is



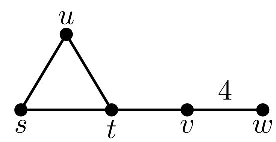
and $\mathcal{D}_{\bullet\bullet}$ is



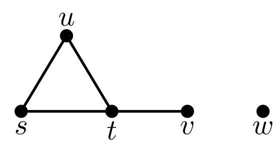
Computing the terms in the right hand side of the isomorphism of Theorem 2.1.6 therefore gives:

$$\begin{aligned} H_2(W;\mathbb{Z}) &= H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2) \\ &= \mathbb{Z}_2 \oplus 0 \oplus 0 \\ &= \mathbb{Z}_2. \end{aligned}$$

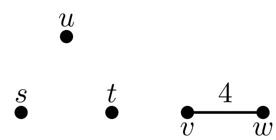
EXAMPLE 2.1.11. Consider the Coxeter group W defined by the following diagram $\mathcal{D}_W$:



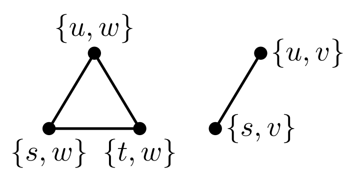
then the diagram $\mathcal{D}_{odd}$ is



the diagram $\mathcal{D}_{even}$ is



and the diagram $\mathcal{D}_{\bullet\bullet}$ is



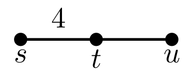
Computing the terms in the right hand side of the isomorphism of Theorem 2.1.6 therefore gives:

$$\begin{aligned} H_2(W;\mathbb{Z}) &= H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2) \\ &= (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

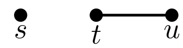
Using similar methods we compute the third homology of a finite rank Coxeter group. In the majority of cases we have a full description for $H_3(W;\mathbb{Z})$, and for a specific type of Coxeter diagram we have the result modulo extensions. The statement of the theorem relies on introducing more diagrams derived from the Coxeter Diagram $\mathcal{D}_W$, described below.

DEFINITION 2.1.12. Suppose W is a finite rank Coxeter group and $\mathcal{D}_W$ is its diagram. We define diagrams that arise from $\mathcal{D}_W$ as follows.

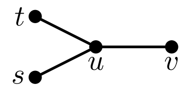
- $\mathcal{D}_{odd}$ is the diagram with vertex set S and an edge between s and t in S if $m(s,t)$ is odd. For example when W is the Coxeter group of type B_3 with diagram



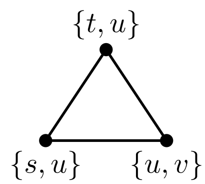
then $\mathcal{D}_{odd}$ is given by



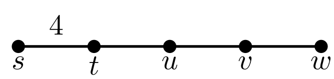
- $\mathcal{D}_{A_2}$ is the diagram with vertex set $\{\{s,t\} \mid s,t \in S,\ m(s,t)=3\}$ i.e. the set of pairs of vertices which appear in an A_2 subdiagram of $\mathcal{D}_W$. There is an edge between $\{s_1,t_1\}$ and $\{s_2,t_2\}$ in $\mathcal{D}_{A_2}$ if $s_1 = s_2$ and $m(t_1,t_2)=2$ i.e. if the two A_2 subdiagrams are not equal, and fit into an A_3 subdiagram of $\mathcal{D}_W$. For example when W is the Coxeter group of type D_4 with diagram



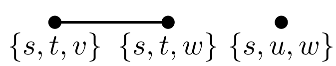
then $\mathcal{D}_{A_2}$ is given by



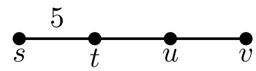
- $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is the diagram with vertex set $\{\{s,t,u\} \mid s,t,u \in S,\ m(s,t)=m(s,u)=2$ and $m(t,u)$ is even$\}$ i.e. the set of triples of vertices which appear in an $A_1 \times I_2(2p)$ subdiagram of $\mathcal{D}_W$. There is an edge between $\{s_1,t_1,u_1\}$ and $\{s_2,t_2,u_2\}$ in $\mathcal{D}_{A_2}$ if $t_1=t_2$, $u_1=u_2$ and and $m(s_1,s_2)$ is odd i.e. if the two $A_1 \times I_2(2p)$ subdiagrams are not equal, and appear as subdiagrams of an $I_2(odd) \times I_2(even)$ subdiagram of $\mathcal{D}_W$. For example when W is the Coxeter group of type B_5 with diagram



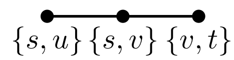
then $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is given by



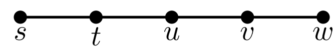
- $\mathcal{D}_{\bullet\bullet}$ is the diagram with vertex set $\{\{s,t\} \mid s,t \in S,\ m(s,t)=2\}$ i.e. the set of pairs of commuting vertices which appear as an $A_1 \times A_1$ subdiagram of $\mathcal{D}_W$. There is an edge between $\{s_1,t_1\}$ and $\{s_2,t_2\}$ in $\mathcal{D}_{\bullet\bullet}$ if $s_1 = s_2$ and $m(t_1,t_2)$ is odd i.e. if the two subdiagrams are not equal, and appear as subdiagrams of an $A_1 \times I_2(odd)$ subdiagram of $\mathcal{D}_W$. For example when W is the Coxeter group of type H_4 with diagram



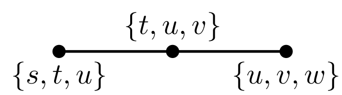
then $\mathcal{D}_{\bullet\bullet}$ is given by



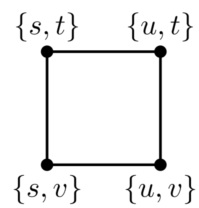
- $\mathcal{D}_{A_3}$ is the diagram with vertex set $\{\{s,t,u\} \mid s,t,u \in S, m(s,t) = m(t,u) = 3 \text{ and } m(s,u) = 2\}$ i.e. the set of triples of vertices which appear in an A_3 subdiagram of $\mathcal{D}_W$. There is an edge between $\{s_1,t_1,u_1\}$ and $\{s_2,t_2,u_2\}$ in $\mathcal{D}_{A_3}$ if $t_1 = s_2$, $u_1 = t_2$ and $m(s_1,u_2) = 2$ i.e. if the two A_3 subdiagrams are not equal, and fit into an A_4 subdiagram of $\mathcal{D}_W$. For example when W is the Coxeter group of type A_5 with diagram



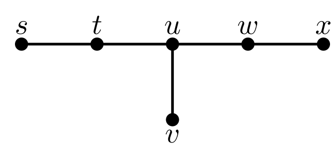
then $\mathcal{D}_{A_3}$ is given by



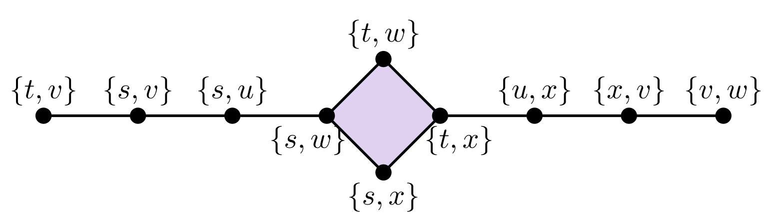
- $\mathcal{D}^{\square}_{\bullet\bullet}$ is the CW complex formed from the diagram $\mathcal{D}_{\bullet\bullet}$ by attaching a $2-$*cell* to every square. Squares in $\mathcal{D}_{\bullet\bullet}$ have the form



For example when W is the Coxeter group of type E_6 with diagram



then $\mathcal{D}^{\square}_{\bullet\bullet}$ is given by



THEOREM 2.1.13. *Given a finite rank Coxeter group W such that $\mathcal{D}_W$ does not have a subdiagram of the form $Y \sqcup A_1$, where Y is a loop in the Coxeter diagram $\mathcal{D}_{odd}$, there is an*

isomorphism

$$\begin{aligned} H_3(W;\mathbb{Z}) &\cong H_0(\mathcal{D}_{odd};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)})\oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2)\\ &\oplus(\bigoplus_{\substack{W(H_3)\subseteq W\\ W(B_3)\subseteq W}}\mathbb{Z}_2)\oplus(H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)\bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2))\end{aligned}$$

where each diagram is as described in Definition 2.1.12, and viewed as a simplicial complex. In this equation, $\bigcirc$ *denotes a known non-trivial extension of* $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ *by* $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ *given by the extension matrix* X_W *defined in Definition 2.5.52.*

If W *is such that* $\mathcal{D}_W$ *has a subdiagram of the form* $Y \sqcup A_1$ *where* Y *is a 1-cycle in the Coxeter diagram* $\mathcal{D}_{odd}$*, then there is an isomorphism modulo extensions*

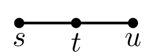
$$\begin{aligned} H_3(W;\mathbb{Z}) &\cong H_0(\mathcal{D}_{odd};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)})\oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2)\\ &\oplus(\bigoplus_{\substack{W(H_3)\subseteq W\\ W(B_3)\subseteq W}}\mathbb{Z}_2)\oplus(H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)\bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2))\\ &\oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2),\end{aligned}$$

where the unknown extensions involve the $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ *summand.*

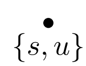
These results arise from computation of the isotropy spectral sequence, which will be introduced in this Chapter, for the Davis complex Σ_W. These computations rely heavily on a free resolution for Coxeter groups as written by De Concini and Salvetti in *Cohomology of Coxeter groups and Artin groups* [**18**]. We give some example computations below.

Example 2.1.14. An example of Theorem 2.1.13 for an infinite Coxeter group can be found in the introduction to this thesis.

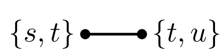
Example 2.1.15. Consider the Coxeter group W of type A_3 defined by the following diagram $\mathcal{D}_W$:



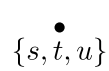
then the diagram $\mathcal{D}_{odd}$ is $\mathcal{D}_W$ and the diagram $\mathcal{D}_{\bullet\bullet}$ is



the diagram $\mathcal{D}_{A_2}$ is



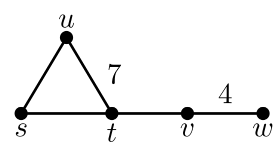
the diagram $\mathcal{D}_{A_3}$ is



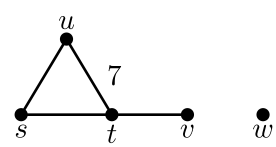
the diagram $\mathcal{D}^{\square}_{\bullet\bullet} = \mathcal{D}_{\bullet\bullet}$ and the diagram $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is the empty diagram. We also note that there are no edges with label greater than 3, and no H_3 or B_3 subdiagrams. We see there is no loop in the diagram $\mathcal{D}_{odd}$ and therefore we are in the first case of the theorem. Computing the terms in the right hand side of the isomorphism of Theorem 2.1.13 therefore gives:

$$\begin{aligned} H_3(W;\mathbb{Z}) &\cong H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \oplus H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \\ &\quad \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \oplus (H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus 0 \oplus 0 \\ &\quad \oplus 0 \oplus (\mathbb{Z}_2 \bigcirc \mathbb{Z}_2) \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \\ &\quad \oplus \mathbb{Z}_4 \\ &= \mathbb{Z}_{12} \oplus \mathbb{Z}_2. \end{aligned}$$

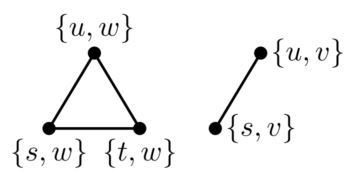
EXAMPLE 2.1.16. Consider the Coxeter group W defined by the following diagram $\mathcal{D}_W$:



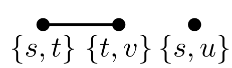
then the diagram $\mathcal{D}_{odd}$ is



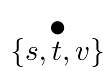
the diagram $\mathcal{D}_{\bullet\bullet}$ is



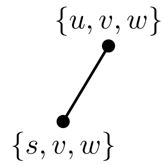
the diagram $\mathcal{D}_{A_2}$ is



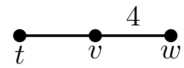
the diagram $\mathcal{D}_{A_3}$ is



the diagram $\mathcal{D}^{\square}_{\bullet\bullet} = \mathcal{D}_{\bullet\bullet}$ and the diagram $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is



We also note that there are two edges with label greater than 3, and one B_3 subdiagram:



We see there is a loop in the diagram $\mathcal{D}_{odd}$ and a vertex disjoint from this loop (w) in $\mathcal{D}_W$, therefore we are in the second case of the theorem. Computing the terms in the right hand side of the isomorphism of Theorem 2.1.13 therefore gives:

$$\begin{aligned}
H_3(W;\mathbb{Z}) \quad &\cong \quad H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \oplus H_0(\mathcal{D}_{\bullet\ \bullet\overset{\text{even}}{—}\bullet};\mathbb{Z}_2) \\
&\quad \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \oplus (H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) \\
&\quad \oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \\
&= \quad (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_7) \oplus (\mathbb{Z}_2) \\
&\quad (\mathbb{Z}_2) \oplus (\mathbb{Z}_2 \bigcirc (\mathbb{Z}_2 \oplus \mathbb{Z}_2)) \\
&\quad \oplus \mathbb{Z}_2 \text{ modulo extensions}
\end{aligned}$$

and here the extension $(\mathbb{Z}_2 \bigcirc (\mathbb{Z}_2 \oplus \mathbb{Z}_2))$ is given by $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

2.2. Outline of proof

We introduce the isotropy spectral sequence in Section 2.3, and specifically describe the spectral sequence for the Davis complex of W, Σ_W, in Section 2.3.14. This spectral sequence abuts to the homology of the Coxeter groups, and in this section we give explicit formulas for the groups on the E^1 page of the spectral sequence, and the d^1 differential, which is given by a transfer map. We also introduce a pairing for the isotropy spectral sequence of the Davis complex in Section 2.3.17, for use later on in the proof. Following this, Section 2.4 proves Theorem 2.1.6 by computing the E^2 page of the spectral sequence on a given diagonal, and feeding these computations into the proof in Section 2.4.11.

For the computation of Theorem 2.1.13 in Section 2.5, much more machinery must be used. In Section 2.5.1 the free resolution for finite Coxeter groups, of De Concini and Salvetti [**18**], is introduced. In order to apply the transfer map to computations using this resolution, a chain map between resolutions is computed in Section 2.5.8. Using these tools, the E^2 page of the spectral sequence on a given diagonal is computed. Following this, Section 2.5.34 proves that all further differentials to and from this diagonal are zero. The possible extension problems arising on the limiting page at this diagonal are treated and discussed in Section 2.5.48 and all of the computations are fed into the proof of Theorem 2.1.13 in Section 2.5.58.

2.3. Introduction to the isotropy spectral sequence

During this chapter we use the spectral sequence associated to a group action on an acyclic CW complex, given by Equation (7.10) in chapter VII of Brown *Cohomology of Groups* [**12**]. In this section we follow Brown to introduce this spectral sequence. We start with a short diversion on extension of scalars and induction.

2.3.1. Induction.

DEFINITION 2.3.2 (see Brown [**12**, III.3]). Given a ring homomorphism $\alpha : R \to S$ and an R module M, we construct the tensor product $S \otimes_R M$ where S is considered an R module via α, i.e. $s \cdot r = s\alpha(r)$. This construction is called *extension of scalars from R to S.*

DEFINITION 2.3.3 (see Brown [**12**, III.5]). Given the ring homomorphism $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$ for H a subgroup of G, extension of scalars is called *induction* from H to G. It is denoted as follows

$$\mathrm{Ind}_H^G M := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

Since the action of H on G is free, we can decompose $\mathrm{Ind}_H^G(M)$ as a sum over left coset representatives of H in G as follows

$$\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M = \bigoplus_{g \in G/H} g \otimes M.$$

where $g \otimes M$ is the set $\{g \otimes m | m \in M\}$, which is isomorphic to M via the map which forgets g. There is a canonical map $i : M \to (\mathbb{Z}G \otimes_{\mathbb{Z}H} M)$ via $i(m) = 1 \otimes M$ and this maps M isomorphically into the $1 \otimes M$ summand of the decomposition. Under the action of G, $g(1 \otimes M) = g \otimes M$ and so we can write each summand in the decomposition as a transform of the canonical M sub-module under the G action. We therefore have the following decomposition [**12**, III.5.1]

$$\mathrm{Ind}_H^G = \bigoplus_{g \in G/H} gM.$$

We are interested in the case where N is a G-module whose underlying abelian group has a decomposition $N = \underset{i \in I}{\oplus} M_i$ over an indexing set I. We require the action of G on N to satisfy that g in G permutes the summands M_i in a way dictated by an action of G on I, and we note that g may also act on the individual summand M_i non-trivially.

PROPOSITION 2.3.4 (see Brown [**12**, III.5.4]). *Suppose N and G are as above. Let G_i be the stabiliser of i in I under the action of G, and let E be a set of orbit representatives. Then M_i is a G_i-module and there is a G-isomorphism $N \cong \bigoplus_{i \in E} \mathrm{Ind}_{G_i}^G M_i$.*

We apply Proposition 2.3.4 to the case where X is a G-CW-complex, following Example III.5.5(b) in Brown [**12**]. In this case the G module $C_n(X)$ can be written as a direct sum of copies of $\mathbb{Z}$. There is one copy of $\mathbb{Z}$, $\mathbb{Z}_\sigma$, for each n-cell of X, σ, and so

$$C_n(X) = \bigoplus_{\sigma\ n-\text{cell of } X} \mathbb{Z}_\sigma.$$

We call $\mathbb{Z}_\sigma$ the *orientation module* for the cell σ. It is the group $\mathbb{Z} = \langle -1, 1\rangle$, with the two generators corresponding to the two orientations of σ. The $\mathbb{Z}_\sigma$ summands of $C_n(X)$ are permuted by G, according to the action of G on the set of n-cells. Let G_σ be the stabiliser of a cell σ under the G action on the n-cells. Then G_σ acts on $\mathbb{Z}_\sigma$ via g acting as $+1$ if g preserves the orientation of σ and -1 otherwise. Letting $\mathcal{O}_n$ be the set of orbit representatives for the action of G on the n-cells, and we apply the proposition to get

$$C_n(X) \cong \bigoplus_{\sigma\in\mathcal{O}_n} \mathrm{Ind}_{G_\sigma}^G \mathbb{Z}_\sigma.$$

We end this section with the statement of Shapiro's Lemma:

PROPOSITION 2.3.5 (Shapiro's Lemma, see Brown [**12**, III.6.2]). *If $H \subseteq G$ is a subgroup of G and M is an H-module then*

$$H_*(H; M) \cong H_*(G; \mathrm{Ind}_H^G M).$$

2.3.6. Spectral sequence of a double complex.

DEFINITION 2.3.7 (see Brown [**12**, VII.3]). A *double complex* is a bi-graded module $(C_{p,q})_{p,q\in\mathbb{Z}}$ with a horizontal differential $\partial^h : C_{p,q} \to C_{(p-1),q}$ and a vertical differential $\partial^v : C_{p,q} \to C_{p,(q-1)}$ such that $\partial^h\partial^v = \partial^v\partial^h$. Given a double complex, the associated *total complex* TC is the chain complex defined by setting

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}$$

and setting the differential to be $\partial|_{C_{p,q}} = \partial^h + (-1)^p\partial^v$.

EXAMPLE 2.3.8. Given two chain complexes C^1 and C^2, one can define the double complex $C_{p,q} = C^1_p \otimes C^2_q$. The associated total complex is then the tensor product of chain complexes $C^1 \otimes C^2$.

DEFINITION 2.3.9 (see Brown [**12**, III.1]). Let G be a group and F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. For a G-module M we define the *group homology of G with coefficients in M* to be

$$H_*(G; M) = H_*(F \otimes_G M).$$

We generalise this by considering a chain complex $C = (C_n)_{n\geq 0}$ of G-modules as coefficients.

DEFINITION 2.3.10 (see Brown [**12**, VII.5]). Let G, F and C as above. Then the *group homology of G with coefficients in C* is given as

$$H_*(G;C) = H_*(F \otimes_G C)$$

where $F \otimes_G C$ is the total complex of the double complex $(F_* \otimes_G C_*)$.

Given a chain complex $C = (C_n)_{n\in\mathbb{Z}}$ and a filtration F_pC which is dimension-wise finite, i.e. $\{F_p(C_n)\}_{p\in\mathbb{Z}}$ is a finite filtration of C_n for each n, there exists a spectral sequence [**12**, VII.2]

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C).$$

Combining this with Definition 2.3.7, we associate two spectral sequences to a double complex. Given a double complex $C = (C_{p,q})_{p,q\in\mathbb{Z}}$ one can filter the total space TC by $F_p((TC)_n) = \bigoplus_{i\leq p} C_{i,n-i}$. This is finite in the case when C is a *first quadrant double complex*, i.e. $C_{p,q}$ is only non-zero for p and q both non-negative integers, and we will deal only with this case. Then we have a spectral sequence with the following properties:

$$E^0_{pq} = C_{p,q} \qquad d^0 = \pm\partial^v \qquad E^1 = H_q(C_{p,*}) \Rightarrow H_{p+q}(TC)$$

where d^1 is the map induced on E^1 by ∂^h.

One can also filter the total space TC by $F_p((TC)_n) = \bigoplus_{j\leq p} C_{n-j,j}$, and this is also finite when C is first quadrant. This gives the spectral sequence with the following properties:

$$E^0_{pq} = C_{p,q} \qquad d^0 = \pm\partial^h \qquad E^1 = H_q(C_{*,p}) \Rightarrow H_{p+q}(TC) \tag{1}$$

where d^1 is the map induced on E^1 by ∂^v. Thus for a double complex there are two spectral sequences which both converge to the homology of the total complex.

We are interested in the specific case of Definition 2.3.10 where the double complex arises from F a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ for a group G and C a positive chain complex. The double complex is therefore first quadrant with the form $(F_p \otimes_G C_q)$ and the second spectral sequence in Equation (1) has the form [**12**, VII.5.(5.3)]

$$E^1_{pq} = H_q(F_* \otimes_G C_p) = H_q(G;C_p) \Rightarrow H_{p+q}(F \otimes_G C) = H_{p+q}(G;C) \tag{2}$$

where the d^1 is the map induced on E^1 by the chain differential $\partial : C_p \to C_{p-1}$.

2.3.11. Equivariant homology and the isotropy spectral sequence. We now follow Section 7 in Chapter VII of Brown [**12**] and apply the previous theory to the study of equivariant homology.

DEFINITION 2.3.12. For G a group and X a G-complex, we define the *equivariant homology groups* of (G, X) to be the homology of G with coefficients in the chain complex $C(X)$ as in Definition 2.3.10. We denote this:

$$H^G_*(X;M) := H_*(G;C(X,M)).$$

In this case Equation (2) gives the following spectral sequence:

$$E^1_{p,q} = H_q(G; C_p(X, M)) \Rightarrow H^G_{p+q}(X; M).$$

Consider now the left hand side. We have the following decomposition for $C_p(X, M)$:

$$C_p(X, M) = C_p(X) \otimes M = \bigoplus_{\sigma \in X_p} \mathbb{Z}_\sigma \otimes M$$

where $\mathbb{Z}_\sigma$ is the orientation module for σ, and X_p is the set of p-cells in X. Letting $M_\sigma = \mathbb{Z}_\sigma \otimes M$ and then applying the results on induction from Section 2.3.1 gives the following decomposition:

$$C_p(X, M) = \bigoplus_{\sigma \in X_p} M_\sigma = \bigoplus_{\sigma \in \mathcal{O}_p} \mathrm{Ind}^G_{G_\sigma} M_\sigma$$

where $\mathcal{O}_p$ is a set of coset representatives for X_p with respect to the G-action.

We may now apply Shapiro's Lemma (Proposition 2.3.5) to the E^1 term of the spectral sequence:

$$\begin{aligned} E^1_{p,q} = H_q(G; C_p(X, M)) &= H_q(G; \bigoplus_{\sigma \in \mathcal{O}_p} \mathrm{Ind}^G_{G_\sigma} M_\sigma) \\ &= \bigoplus_{\sigma \in \mathcal{O}_p} H_q(G_\sigma; M_\sigma) \end{aligned}$$

so the spectral sequence has the form:

$$E^1_{p,q} = \bigoplus_{\sigma \in \mathcal{O}_p} H_q(G_\sigma; M_\sigma) \Rightarrow H^G_{p+q}(X; M).$$

We finish with the observation from Brown that should X be acyclic, we have

$$H^G_*(X; M) \cong H_*(G; M),$$

which gives the spectral sequence the form

$$E^1_{p,q} = \bigoplus_{\sigma \in \mathcal{O}_p} H_q(G_\sigma; M_\sigma) \Rightarrow H_{p+q}(G; M). \tag{3}$$

We let this spectral sequence be called the *isotropy spectral sequence*.

We now discuss the d^1 differential for the isotropy spectral sequence, following Brown [**12**, VII.8]. Consider the following diagram:

$$\begin{tikzcd}
E^1_{p,q} \arrow[rr, "d^1"] \arrow[d, equal] & & E^1_{p-1,q} \arrow[d, equal] \\
H_q(G; C_p(X, M)) \arrow[rr, "H_q(G;\partial)"] \arrow[d, "\cong"] & & H_q(G; C_{p-1}(X, M)) \arrow[d, "\cong"] \\
\bigoplus_{\sigma \in \mathcal{O}_p} H_q(G_\sigma; M_\sigma) \arrow[rr, "\phi"] & & \bigoplus_{\sigma \in \mathcal{O}_{p-1}} H_q(G_\sigma; M_\sigma).
\end{tikzcd}$$

Here the central map from left to right is given by the fact that in Equation (2) the differential on the E^1 page is induced by the chain complex differential $\partial : C_p(X, M) \to C_{p-1}(X, M)$. We will define a map ϕ on the bottom row such that under the vertical isomorphism, the map ϕ gives the d^1 differential (see Brown [**12**, VII.8.1]). We define ϕ in three stages

(1) Consider a p-cell σ and a $(p-1)$-cell τ of X. Denote by $\partial_{\sigma\tau}$ the component of the differential $\partial : C_p(X, M) \to C_{p-1}(X, M)$ restricted to σ in the source and τ in the image. Recall that $C_p(X, M)$ is a sum of modules M_σ for every p-cell σ and so $\partial_{\sigma\tau} : M_\sigma \to M_\tau$. Let $\mathcal{F}_\sigma = \{\tau \mid \partial_{\sigma\tau} \neq 0\}$. This corresponds to $(p-1)$ cells in the boundary of the p-cell σ. Then since G_σ is the stabilizer of σ, the set $\mathcal{F}_\sigma$ is G_σ-invariant. Let $G_{\sigma\tau} = G_\sigma \cap G_\tau$. Then when τ is in $\mathcal{F}_\sigma$ the index of $G_{\sigma\tau}$ in G_σ is finite. We can therefore define a transfer map
$$t_{\sigma\tau} : H_q(G_\sigma; M_\sigma) \to H_q(G_{\sigma\tau}; M_\sigma).$$

(2) Since ∂ is G-equivariant, it follows that $\partial_{\sigma\tau} : M_\sigma \to M_\tau$ is $G_{\sigma\tau}$-equivariant. Together with the inclusion $G_{\sigma\tau} \hookrightarrow G_\tau$ this induces a map
$$u_{\sigma\tau} : H_q(G_{\sigma\tau}; M_\sigma) \to H_q(G_\tau; M_\tau).$$

(3) Under the isomorphism from the central to the bottom row of the diagram, we are taking a sum over orbit representatives. It may be that $H_q(G_\tau; M_\tau)$ is not a summand on the E^1 page, if τ is not a chosen orbit representative. Let τ_0 be the orbit representative for the G-orbit of τ (in $\mathcal{O}_{p-1}$), and choose $g(\tau)$ in G such that $g(\tau)\tau = \tau_0$. Then there is an isomorphism $M_\tau \to M_{\tau_0}$ given by the action of $g(\tau)$ on $C_{p-1}(X, M)$ and this is compatible with the conjugation isomorphism $G_\tau \to G_{\tau_0}$ given by conjugating by $g(\tau)$. Together these give an isomorphism
$$v_\tau : H_q(G_\tau; M_\tau) \to H_q(G_{\tau_0}; M_{\tau_0}).$$

Definition 2.3.13. Given the maps described above, the d^1 differential of the isotropy spectral sequence is
$$\phi : \bigoplus_{\sigma \in \mathcal{O}_p} H_q(G_\sigma; M_\sigma) \to \bigoplus_{\sigma \in \mathcal{O}_{p-1}} H_q(G_\sigma; M_\sigma)$$

when on each summand of the left hand side we define ϕ to be

$$\phi \restriction_{H_q(G_\sigma;M_\sigma)} = \sum_{\tau \in \mathcal{F}'_\sigma} v_\tau u_{\sigma\tau} t_{\sigma\tau}$$

where $\mathcal{F}'_\sigma$ is the set of representative for the orbits of the cells in $\mathcal{F}_\sigma / G_\sigma$.

2.3.14. Isotropy spectral sequence for the Davis Complex. We now apply the isotropy spectral sequence in the case that the group is a Coxeter group W with generating set S, the coefficient module is the integers $\mathbb{Z}$ and the W-CW-complex is the Davis complex Σ_W (introduced in Section 1.3).

Recall that the Davis complex is contractible (Proposition 1.3.12) and hence acyclic. Then Equation (3) becomes

$$E^1_{p,q} = \bigoplus_{\sigma \in \mathcal{O}_p} H_q(W_\sigma; \mathbb{Z}_\sigma) \Rightarrow H_{p+q}(W; \mathbb{Z}),$$

since $\mathbb{Z}_\sigma \otimes \mathbb{Z} \cong \mathbb{Z}_\sigma$.

Recall that each p-cell of Σ_W is represented by a spherical coset wW_T where T has size p, and the vertices of the cell are given by the set $\{wW_\emptyset | w \in wW_T\}$. W acts by left multiplication and so we can choose the orbit representatives of p-cells to be the cosets eW_T where T has size p. Recall that $\mathcal{S}$ is the set $\{T \subset S \mid W_T \text{ is finite}\}$. Hence the set of orbit representatives $\mathcal{O}_p$ is given by spherical subgroups W_T with T in $\mathcal{S}$ having size p. The stabiliser of a cell represented by a spherical subgroup W_T under the W-action is W_T itself, since the action of W is given by left multiplication. Every member of the generating set T of W_T acts on the cell by reflection and therefore reverses the orientation of the cell. The action of an element of W_T on the orientation module will therefore be the identity if the element has even length, or negation if the element has odd length. Under these choices, the isotropy spectral sequence becomes

$$E^1_{p,q} = \bigoplus_{\substack{T \in \mathcal{S} \\ |T| = p}} H_q(W_T; \mathbb{Z}_T) \Rightarrow H_{p+q}(W; \mathbb{Z})$$

where we write $\mathbb{Z}_T$ as the orientation module for the cell corresponding to W_T. Putting this together we get E^1 page as shown in Figure 1.

$$
\begin{array}{c|ccccccccc}
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
3 & H_3(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_3(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=2}} H_3(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=3}} H_3(W_T;\mathbb{Z}_T) & & \cdots \\
2 & H_2(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_2(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=2}} H_2(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=3}} H_2(W_T;\mathbb{Z}_T) & & \cdots \\
1 & H_1(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_1(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=2}} H_1(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=3}} H_1(W_T;\mathbb{Z}_T) & & \cdots \\
0 & H_0(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_0(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=2}} H_0(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=3}} H_0(W_T;\mathbb{Z}_T) & & \cdots \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4
\end{array}
$$

FIGURE 1. The E^1 page of the isotropy spectral sequence for the Davis complex

Here the zeroth column only has one summand, since only the empty set satisfies the criteria of generating a spherical subgroup and having size zero. In the first column, we note that all generators in S generate a cyclic group of order two, which is finite and so we sum over all t in S. The horizontal d^1 maps are defined by applying the definition of the d^1 differential for the isotropy spectral sequence (Definition 2.3.13) in the specific case for the Davis complex Σ_W.

PROPOSITION 2.3.15. *In the isotropy spectral sequence for the Davis complex Σ_W, denote the d^1 differential component restricted to the $H_q(W_T;\mathbb{Z}_T)$ component in the source and the $H_q(W_U;\mathbb{Z}_U)$ in the target by $d^1_{T,U}$. Then this map is non zero only when $U \subset T$ and is given by the following transfer map:*

$$d^1_{T,U} : H_q(W_T;\mathbb{Z}_T) \quad \to \quad H_q(W_U;\mathbb{Z}_U).$$

On the chain level we compute $H_q(W_T;\mathbb{Z}_T)$ by computing homology of $\mathbb{Z}_T \otimes_{W_T} F_{W_T}$ for F_{W_T} a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}W_T$. To define the transfer map we compute $H_q(W_U;\mathbb{Z}_U)$ by computing homology of $\mathbb{Z}_U \otimes_{W_U} F_{W_T}$ for F_{W_T} again a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}W_T$. The transfer map can then be defined on the chain level by the map below, where $m \otimes x$ is in

$\mathbb{Z}_T \otimes F_{W_T}$ and $W_U \backslash W_T$ is a set of orbit representatives for W_U in W_T.

$$d^1_{T,U} : m \otimes x \quad \mapsto \quad \sum_{g \in W_U \backslash W_T} m \cdot g^{-1} \otimes g \cdot x.$$

Proof. Consider the three maps of Definition 2.3.13. Recall that an orbit representative for a p-cell is given by eW_T with T in $\mathcal{S}$ and $|T| = p$. The set $\mathcal{F}_T = \{U \mid \partial_{T,U} \neq 0\}$ is then given by cosets wW_U with $|U| = (p-1)$ such that $wW_U \subset W_T$, which is satisfied if and only if $U \subset T$ and $w \in W_T$ by Lemma 1.3.2. Since W_T is the stabiliser of the cell eW_T, this gives that the orbit set $(\{U \mid \partial_{T,U} \neq 0\}/W_T)$ is given by $\{U \mid |U| = p-1\,,\, U \subset T\}$. Since these are already in the set of orbit representatives of $(p-1)$-cells we have $\mathcal{F}'_T = \{U \mid |U| = p-1\,,\, U \subset T\}$ and so the map ϕ restricted to the $H_q(W_T; \mathbb{Z}_T)$ summand maps only to summands $H_q(W_U; \mathbb{Z}_U)$ when $U \subset T$. In other words, this gives that the isomorphism v_τ in the definition of ϕ is the identity map in this case, since the map v_τ maps between $(p-1)$-cells and their orbit representatives and in this case the $(p-1)$-cells we consider are already the orbit representatives. The intersection $Stab(W_T) \cap Stab(W_U) = W_T \cap W_U = W_U$ and the action of W_U on $\mathbb{Z}_T$ is precisely the action of W_U on $\mathbb{Z}_U$. Therefore the map $u_{\sigma\tau}$ in the definition of ϕ is also an isomorphism and it follows that

$$\begin{aligned} \phi \restriction_{H_q(G_\sigma, M_\sigma)} &= \sum_{\tau \in \mathcal{F}'_\sigma} v_\tau u_{\sigma\tau} t_{\sigma\tau} \\ \phi \restriction_{H_q(W_T; \mathbb{Z}_T)} &= \sum_{U \in \mathcal{F}'_T} t_{T,U} \end{aligned}$$

where $t_{T,U}$ is the transfer map

$$t_{T,U} : H_q(W_T; \mathbb{Z}_T) \to H_q(W_U; \mathbb{Z}_U).$$

Note that cycles in $H_q(W_T; \mathbb{Z}_T)$ are represented by chains in $\mathbb{Z}_T \otimes F_{W_T}$ where F_{W_T} is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}_{W_T}$. Letting $m \otimes x$ be an element on the chain level yields the formula, where the transfer map on the chain level is computed via Brown [**12**, III.9]. $\square$

Since we are interested in $H_2(W; \mathbb{Z})$ and $H_3(W; \mathbb{Z})$ we wish to consider the groups on the red diagonal of Figure 1 at E^∞ for H_2 and the blue diagonal of Figure 1 for H_3. We are summing over finite Coxeter groups with generating set a certain size, and the classification of finite Coxeter groups from Theorem 1.1.12 provides a finite selection of possible groups for each size of generating set. Therefore there is a finite number of calculations to do in order to find an E^1 term in general.

Lemma 2.3.16. *Let $V \hookrightarrow W$ be an inclusion of Coxeter groups satisfying that V is parabolic i.e. that the generating set for V, S_V, is a subset of the generating set for W, S and $\mathcal{D}_V$ is a full subdiagram of $\mathcal{D}_W$. Then there is a map of isotropy spectral sequences*

$$E(V) \to E(W)$$

which is an inclusion on the E^1 page.

PROOF. The inclusion $j : V \hookrightarrow W$ induces an inclusion $W_V\mathcal{S}_V \subset W\mathcal{S}$, since S_V is a subset of S and therefore $\mathcal{S}_V$ is a subset of $\mathcal{S}$. This induces a map between the realisations $i : \Sigma_V \hookrightarrow \Sigma_W$, and therefore a map between the chains on p-cells $C_p(\Sigma_V, \mathbb{Z}) \xrightarrow{i_*} C_p(\Sigma_W, \mathbb{Z})$. We therefore have the following diagram:

$$\begin{tikzcd}
E^1_{p,q}(V) \arrow[r] \arrow[d, equal] & E^1_{p,q}(W) \arrow[d, equal] \\
H_q(V; C_p(\Sigma_V, \mathbb{Z})) \arrow[r, "H_q(j_*;i_*)"] \arrow[d, "\cong"] & H_q(W; C_p(\Sigma_W, \mathbb{Z})) \arrow[d, "\cong"] \\
\bigoplus_{\substack{U\in\mathcal{S}_V \\ |U|=p}} H_q(W_U; \mathbb{Z}_U) \arrow[r, dotted] & \bigoplus_{\substack{T\in\mathcal{S} \\ |T|=p}} H_q(W_T; \mathbb{Z}_T)
\end{tikzcd}$$

where the dotted map is induced by the map on p-cells on the central row. Every spherical subgroup of V will also be a spherical subgroup of W, since it is a full inclusion, and this will correspond to a map between the p-cells representing these spherical subgroups. We therefore have

$$\begin{array}{rcl}
E^1_{p,q}(V) & \hookrightarrow & E^1_{p,q}(W) \\
\bigoplus\limits_{\substack{U\in\mathcal{S}_V \\ |U|=p}} H_q(W_U; \mathbb{Z}_U) & \hookrightarrow & \bigoplus\limits_{\substack{T\in\mathcal{S} \\ |T|=p}} H_q(W_T; \mathbb{Z}_T).
\end{array}$$

Since the d^1 differential is defined via the transfer map on each summand, all d^1 differentials in $E(V)$ will map under the inclusion to the same differential in $E(W)$. The inclusion on the E^1 page therefore induces a map of spectral sequences on further pages. This completes the proof. $\square$

2.3.17. Pairings on the isotropy spectral sequence. We now consider a pairing of spectral sequences, for use in Section 2.5.34. We follow May's *A Primer on Spectral Sequences* [**37**] and recall Section 4 on products. For filtered complexes A, B and C, if a pairing

$$A \otimes B \to C$$

is a morphism of filtered complexes, i.e. if $F_pA \cdot F_qB \subset F_{p+q}C$, then this induces a morphism of spectral sequences

$$E^r(A \otimes B) \to E^r(C).$$

Combining this with the Künneth map $E^r(A) \otimes E^r(B) \to E^r(A \otimes B)$ (which is induced by the Künneth map on homology on the E^1 page) defines a pairing

$$\phi : E^r(A) \otimes E^r(B) \to E^r(C)$$

which satisfies a Leibniz formula for differentials, i.e. for x in $E^r(A)$ and y in $E^r(B)$ the pairing satisfies

$$d^r_C(\phi(x \otimes y)) = \phi(d^r_A(x) \otimes y) + (-1)^{deg(x)}\phi(x \otimes d^r_B(y)).$$

Consider the product of two finite Coxeter groups W_U and W_V. Then $W_U \times W_V = W_X$ for $X = U \sqcup V$ as in Section 1.2. For the following notation let W_I be the Coxeter group corresponding to $I \in \{V, U, X\}$. Let S_I be the generating set of W_I and let $\mathcal{S}_I$ be $\mathcal{S}$ for the Coxeter system (W_I, I) (see Definition 1.2.11). Let Σ_I be the Davis complex Σ_{W_I} and F^I be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}W_I$. Let $E(I)$ denote the isotropy spectral sequence for W_I. Then $E(I)$ is the spectral sequence related to the double complex $F^I \otimes C(\Sigma_I, \mathbb{Z})$. Denote the double complex by $I_{p,q}$ and the associated total complex TI. Then $(TI)_n = \bigoplus_{p+q=n} I_{p,q}$ and for the spectral sequence $E(I)$ the total space TI is given the filtration $F_p((TI)_n) = \bigoplus_{i \leq p} I_{n-i,i}$.

Lemma 2.3.18. *The product map $W_U \times W_V \to W_X$ defines a map on chain complexes*

$$C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z}).$$

Proof. With notation as above, the product map induces a map of posets

$$\begin{array}{rcl} W_U\mathcal{S}_U \times W_V\mathcal{S}_V & \to & W_X\mathcal{S}_X \\ (uW_{T_U}, vW_{T_V}) & \mapsto & uv(W_{T_U \sqcup T_V}). \end{array}$$

This in turn induces a map on their realisations

$$\Sigma_U \times \Sigma_V \to \Sigma_X,$$

which is the map that gives the decomposition $\Sigma_X = \Sigma_U \times \Sigma_V$ from Lemma 1.3.13. Consider $C_i(\Sigma_I, \mathbb{Z})$ and note that p-cells of Σ_I are represented by cosets wW_T where $T \in \mathcal{S}_I$. Given an i-cell of Σ_U represented by uW_{T_1} and a j-cell of Σ_V represented by vW_{T_2} we use the above poset map and define an $(i+j)$-cell of Σ_X represented by $uvW_{T_1 \sqcup T_2}$. This gives a pairing $C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z})$. ☐

Theorem 2.3.19. *With the above notation, we can apply the hypothesis of May* [**37**, Section 4] *(that we have a morphism of filtered complexes) and conclude that there is a pairing*

$$\Phi : E^r(U) \otimes E^r(V) \to E^r(X)$$

under which the differentials satisfy a Leibniz formula. Under the decomposition on the E^1 page of the spectral sequence (Figure 1)

$$E^1_{p,q}(I) = H_q(F^I_* \otimes_{W_I} C_p(\Sigma_I, \mathbb{Z})) \cong \bigoplus_{\substack{\bar{I} \in \mathcal{S}_I \\ |\bar{I}| = p}} H_q(W_{\bar{I}}; \mathbb{Z}_{\bar{I}})$$

this pairing induces a pairing Φ_*, which is given by the Künneth map when restricted to individual summands

$$\Phi_* : H_q(W_{\bar{U}}; \mathbb{Z}_{\bar{U}}) \otimes H_{q'}(W_{\bar{V}}; \mathbb{Z}_{\bar{V}}) \xrightarrow{\times} H_{q+q'}(W_{\bar{U}} \times W_{\bar{V}}; \mathbb{Z}_{\bar{U}} \otimes \mathbb{Z}_{\bar{V}}) \xrightarrow{\cong} H_{q+q'}(W_{\bar{X}}; \mathbb{Z}_{\bar{X}})$$

and it follows that the differentials in the isotropy spectral sequence for the Davis complex satisfy a Leibniz formula with respect to the pairing Φ_*.

Proof. To show that Φ is a pairing we must show that the map

$$TU \otimes TV \to TX$$

is a morphism of filtered complexes. We have on the nth-chain level that

$$F_p((TI)_n) = \bigoplus_{i \le p} I_{n-i,i} = \bigoplus_{i \le p} F^I_{n-i} \otimes C_i(\Sigma_I, \mathbb{Z})$$

for I in $\{U, V, X\}$. Since W_U and W_V are subgroups of W_X such that $W_U \times W_V = W_X$, there is a pairing from $F^U_k \otimes F^V_l \to F^X_{k+l}$ (for example by taking $F^X = F^U \otimes F^V$ by Brown [**12**, V.1.1]). Putting this together with the pairing $C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z})$ from the previous lemma gives

$$F_p(TU) \cdot F_q(TV) \subset F_{p+q}(TX)$$

as required in [**37**]. We now consider this pairing under the decomposition on the E^1 page of the isotropy spectral sequence for a Coxeter group W_I, shown in Figure 1:

$$E^1_{p,q}(I) = H_q(F^I_* \otimes_{W_I} C_p(\Sigma_I, \mathbb{Z})) \cong \bigoplus_{\substack{\bar{I} \in \mathcal{S}_I \\ |\bar{I}| = p}} H_q(W_{\bar{I}}; \mathbb{Z}_{\bar{I}})$$

and described in Section 2.3.14. Under this decomposition the map from a single summand on the right of the isomorphism, to the left of the isomorphism, is given by the following map ι_*, induced by ι:

$$F^T_* \otimes_{W_T} C_p(\Sigma_T, \mathbb{Z}_T) \xrightarrow{\quad\iota\quad} F^W_* \otimes_W C_p(\Sigma_W, \mathbb{Z})$$

$$\begin{array}{ccc} H_q(F^T_* \otimes_{W_T} C_p(\Sigma_T, \mathbb{Z}_T)) & \xrightarrow{\iota_*} & H_q(F^W_* \otimes_W C_p(\Sigma_W, \mathbb{Z})) \\ \| & & \| \\ H_q(W_T; \mathbb{Z}_T) & \longrightarrow & H_q(F^W_* \otimes_W C_p(\Sigma_W, \mathbb{Z})). \end{array}$$

If a Coxeter group W_X arises as a product $W_X = W_U \times W_V$ then the pairing Φ, along with the decomposition for each group W_U, W_V, and W_X gives the following diagram

$$\begin{tikzcd}
(E^1_{p,q}(U) \arrow[d, equal] & \otimes & E^1_{p',q'}(V)) \arrow[d, equal] \arrow[r, "\Phi"] & E^1_{(p+p'),(q+q')}(X) \arrow[d, equal] \\
(H_q(F^U_* \otimes_{W_U} C_p(\Sigma_U,\mathbb{Z})) & \otimes & H_{q'}(F^V_* \otimes_{W_V} C_{p'}(\Sigma_V,\mathbb{Z}))) \arrow[r, "\Phi"] & H_{q+q'}(F^X_* \otimes_{W_X} C_{p+p'}(\Sigma_X,\mathbb{Z})) \\
(\bigoplus\limits_{\substack{\bar U\in\mathcal{S}_U\\ |\bar U|=p}} H_q(W_{\bar U};\mathbb{Z}_{\bar U}) \arrow[u, "\oplus\iota_*", "\cong"'] & \otimes & \bigoplus\limits_{\substack{\bar V\in\mathcal{S}_V\\ |\bar V|=p'}} H_{q'}(W_{\bar V};\mathbb{Z}_{\bar V})) \arrow[u, "\oplus\iota_*", "\cong"'] \arrow[r, dotted, "\Phi_*"] & \bigoplus\limits_{\substack{\bar X\in\mathcal{S}_X\\ |\bar X|=p+p'}} H_{q+q'}(W_{\bar X};\mathbb{Z}_{\bar X}). \arrow[u, "\oplus\iota_*", "\cong"']
\end{tikzcd}$$

The map Φ_* is then induced by Φ and the two vertical isomorphisms. The isomorphisms are induced by the component-wise inclusions given by ι_* on each summand. Since the pairing Φ is defined by the pairings $F^U_k \otimes F^V_l \to F^X_{k+l}$ and $C_i(\Sigma_U,\mathbb{Z}) \otimes C_j(\Sigma_V,\mathbb{Z}) \to C_{i+j}(\Sigma_X,\mathbb{Z})$, then component wise, the map Φ_* is given on each summand of
$\bigoplus\limits_{\substack{\bar U\in\mathcal{S}_U\\ |\bar U|=p}} H_q(W_{\bar U};\mathbb{Z}_{\bar U})$ and $\bigoplus\limits_{\substack{\bar V\in\mathcal{S}_V\\ |\bar V|=p'}} H_{q'}(W_{\bar V};\mathbb{Z}_{\bar V})$ by the composite

$$H_q(W_{\bar U};\mathbb{Z}_{\bar U}) \otimes H_{q'}(W_{\bar V};\mathbb{Z}_{\bar V}) \xrightarrow{\times} H_{q+q'}(W_{\bar U} \times W_{\bar V};\mathbb{Z}_{\bar U} \otimes \mathbb{Z}_{\bar V}) \xrightarrow{\cong} H_{q+q'}(W_{\bar X};\mathbb{Z}_{\bar X})$$

where here $\bar X$ is defined such that $W_{\bar U} \times W_{\bar V} = W_{\bar X}$. Here the first map is given by the homology cross product (see [**12**, V.3]), and the second map is given by the fact that if $W_{\bar U} \times W_{\bar V} = W_{\bar X}$ then the orientation modules satisfy $\mathbb{Z}_{\bar U} \otimes \mathbb{Z}_{\bar V} \cong \mathbb{Z}_{\bar X}$. This map is precisely the Künneth map on homology. Extending this component wise definition to a definition on the tensor product of the summations, gives the map Φ_* that lifts to the map Φ on the top row.

This pairing on the decomposition at the E^1 page of the isotropy spectral sequence for the Davis complex will therefore induce a pairing on the E^r page and it follows that the differentials in the isotropy spectral sequence for the Davis complex satisfy a Leibniz property with respect to the pairing Φ_*. ☐

2.4. Calculation for $H_2(W;\mathbb{Z})$

From Section 2.3.14, we have a spectral sequence with E^1 page the following

$$
\begin{array}{c|ccccccc}
\vdots & \vdots & & \vdots & & \vdots & & \vdots \quad \vdots \\
3 & \textcolor{teal}{H_3(W_\emptyset;\mathbb{Z}_\emptyset)} & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_3(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_3(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_3(W_T;\mathbb{Z}_T) \quad \cdots \\
2 & \textcolor{red}{H_2(W_\emptyset;\mathbb{Z}_\emptyset)} & \xleftarrow{d^1} & \textcolor{teal}{\bigoplus\limits_{t\in S} H_2(W_t;\mathbb{Z}_t)} & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_2(W_T;\mathbb{Z}_T) \quad \cdots \\
1 & H_1(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \textcolor{red}{\bigoplus\limits_{t\in S} H_1(W_t;\mathbb{Z}_t)} & \xleftarrow{d^1} & \textcolor{teal}{\bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T)} & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) \quad \cdots \\
0 & H_0(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_0(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \textcolor{red}{\bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T)} & \xleftarrow{d^1} & \textcolor{teal}{\bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T)} \quad \cdots \\
\hline
 & 0 & & 1 & & 2 & & 3 \qquad 4
\end{array}
$$

and the E^∞ page will give us filtration quotients of $H_2(W;\mathbb{Z})$ on the red diagonal. In this section we compute the red diagonal on the E^2 page and note that no further differentials map from non zero groups onto this diagonal. The E^2 computation therefore gives us the limiting groups on the red diagonal and the result follows.

2.4.1. Homology at $E^1_{0,2}$. The $E^1_{0,2}$ term is given by $H_2(W_\emptyset;\mathbb{Z}_\emptyset)$. From Definition 1.1.2, $W_\emptyset$ is the Coxeter group with no generators, i.e. the trivial group, and so $H_*(W_\emptyset;\mathbb{Z}_\emptyset)$ is zero for $* > 0$. Hence $E^1_{0,2}$ is zero, and so $E^2_{0,2}$ and $E^\infty_{0,2}$ are zero.

2.4.2. Homology at $E^1_{1,1}$. The $E^1_{1,1}$ term is given by

$$E^1_{1,1} = \bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t).$$

Each individual summand $H_1(W_t;\mathbb{Z}_t)$ is the homology of the group W_t, i.e. the Coxeter group with single generator t and relation $t^2 = e$ (i.e. the finite Coxeter group $W(A_1)$). Hence we are considering the homology of a cyclic group of order 2, with coefficients in a $\mathbb{Z}_t$ module given by the integers with action where the non-trivial group element t acts on $\mathbb{Z}_t$ by negation.

Lemma 2.4.3. *With notation as above,*

$$H_1(W_t;\mathbb{Z}_t) = 0.$$

Proof. This follows from taking the standard projective resolution for a cyclic group of order 2, tensoring with the coefficient module and calculating homology. It also follows from the resolution introduced later in Section 2.5.1 and in particular Example 2.5.6. $\square$

Since $E^1_{1,1}$ is a sum of groups which are all zero, we conclude that $E^1_{1,1} = \bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t)$ is zero, and hence $E^2_{1,1}$ and $E^\infty_{1,1}$ are zero.

2.4.4. Homology at $E^1_{2,0}$. We finally consider the homology at $E^1_{2,0}$, which is given by

$$E^1_{2,0} = \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=2}} H_0(W_T;\mathbb{Z}_T).$$

Since the other groups on the red diagonal in the spectral sequence are zero, this will be the only contributing group to the red diagonal on the E^∞ page. We start by computing $E^2_{2,0}$, which is given by the homology of the following sequence

$$\bigoplus_{t\in\mathcal{S}} H_0(W_t;\mathbb{Z}_t) \xleftarrow[d^1]{} \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=2}} H_0(W_T;\mathbb{Z}_T) \xleftarrow[d^1]{} \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=3}} H_0(W_T;\mathbb{Z}_T).$$

Recall that the d^1 differential is given by the transfer map defined in Proposition 2.3.15, where the transfer map restricted to the summand corresponding to a spherical subgroup W_T maps into summands corresponding to the spherical subgroup W_U, only when U is a subset of T, and this map is given on the chain level by:

$$\begin{aligned} d^1_{T,U} : H_q(W_T;\mathbb{Z}_T) &\to H_q(W_U;\mathbb{Z}_U) \\ m\otimes x &\mapsto \sum_{g\in W_U\backslash W_T} m\cdot g^{-1}\otimes g\cdot x. \end{aligned}$$

LEMMA 2.4.5. *For all T in $\mathcal{S}$, such that $|T|>0$,*

$$H_0(W_T;\mathbb{Z}_T) = \mathbb{Z}_2.$$

PROOF. This follows from the definition of group homology with coefficients in a module, see [**12**, III.1.(1.5)]. The zeroth homology is given by the coinvariants of the module under the group action:

$$\begin{aligned} H_0(G;M) &= M_G \\ &= \mathbb{Z}\otimes_{\mathbb{Z}G} M. \end{aligned}$$

Since in our case the module is the integers and each group generator acts as multiplication by -1 we compute homology to be the group $\mathbb{Z}_2$. ☐

LEMMA 2.4.6. *Applying the definition of the transfer map for the bottom ($H_0(W_T;\mathbb{Z}_T)$) row of the spectral sequence, and letting the generator of $H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2$ be denoted by 1_T gives the following map, when T' is a subset of T.*

$$\begin{aligned} d^1_{T,T'} : H_0(W_T;\mathbb{Z}_T) &\to H_0(W_{T'};\mathbb{Z}_{T'}) \\ \mathbb{Z}_2 &\to \mathbb{Z}_2 \\ 1_T &\mapsto \begin{cases} 0 & \text{if } |W_T|/|W_{T'}| \text{ is even} \\ 1_{T'} & \text{if } |W_T|/|W_{T'}| \text{ is odd.} \end{cases} \end{aligned}$$

PROOF. From Brown [**12**, III.9.(B)] we know for H a subgroup of G, the transfer map acts on coinvariants as

$$\begin{array}{rcl} tr : M_G & \to & M_H \\ \overline{m} & \mapsto & \displaystyle\sum_{g \in H\backslash G} \overline{\overline{gm}} \end{array}$$

where $\overline{m}$ and $\overline{\overline{m}}$ denote the image of m in M_G, or M_H. In our case this gives

$$\begin{array}{rcl} d^1_{T,T'} : H_0(W_T;\mathbb{Z}_T) & \to & H_0(W_{T'};\mathbb{Z}_{T'}) \\ \mathbb{Z}_2 & \to & \mathbb{Z}_2 \\ 1_T & \mapsto & \displaystyle\sum_{g \in W_{T'}\backslash W_T} 1_{T'} \end{array}$$

since $g \cdot 1 = \pm 1$ is in the class of 1 in $\mathbb{Z}_{T'}/W_{T'}$. Noting that we are mapping into a $\mathbb{Z}_2$ and the number of entries in the sum is $|W_T|/|W_{T'}|$ completes the proof. $\square$

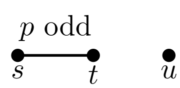
For $X \in \mathcal{S}$, let 1_X be the generator for the summand $H_0(W_X, \mathbb{Z}_X)$ in $\bigoplus_{T \in \mathcal{S}} H_0(W_T;\mathbb{Z}_T)$.

LEMMA 2.4.7. *With notation as above, when T' has size 1 and $T = \{s,t\}$ has size 2 the transfer map d^1 restricted to the T summand is given by*

$$d^1 \restriction_{H_0(W_T;\mathbb{Z}_T)} (1_T) = \begin{cases} 1_s + 1_t & \textit{if } m(s,t) \textit{ odd} \\ 0 & \textit{if } m(s,t) \textit{ even.} \end{cases}$$

PROOF. Note that $|W_x|$ is 2 for all $x \in S$ and since $W_{\{s,t\}}$ is a dihedral group, $|W_{\{s,t\}}|$ is $2 \times m(s,t)$. Then $|W_{\{s,t\}}|/|W_x| = m(s,t)$ for $x \in \{s,t\}$, and we apply Lemma 2.4.6 to compute the differential. $\square$

DEFINITION 2.4.8. We say that a Coxeter group with generating set $T = \{s,t,u\}$ is *of type X* if the Coxeter diagram has the form:

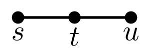
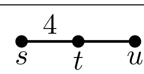
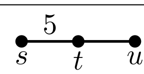
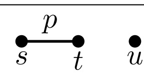


i.e. if $W_T = W(I_2(p)) \times W(A_1)$ and p is odd.

LEMMA 2.4.9. *If T' has size 2 and $T = \{s,t,u\}$ has size 3 the transfer map d^1 restricted to the T summand is given by*

$$d^1 \restriction_{H_0(W_T;\mathbb{Z}_T)} (1_T) = \begin{cases} 1_{\{s,u\}} + 1_{\{t,u\}} & \textit{if } W_T \textit{ is of type } X \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. When $T = \{s,t,u\}$ and W_T is finite, there are a finite number of Coxeter diagrams that may represent W_T, given by groups and products of groups in the classification of finite Coxeter groups (Theorem 1.1.12). The order of these groups and their size two subgroups is documented in the table below, where we recall that $W(A_1)\times W(A_1)\times W(A_1) = W(I_2(2))\times W(A_1)$ and so this group is included in the final case.

W_T	$\mathcal{D}_W$	$\lvert W_T\rvert$	$\lvert W_{\{s,t\}}\rvert$	$\lvert W_{\{s,u\}}\rvert$	$\lvert W_{\{t,u\}}\rvert$
$W(A_3)$		24	6	4	6
$W(D_3)$		48	8	4	6
$W(H_3)$		120	10	4	6
$W(I_2(p))\times W(A_1)$		4p	2p	4	4

Calculating $|W_T|/|W_{T'}|$ in each of these cases therefore gives an even answer (and hence a zero transfer map) unless we are in the final case $W(I_2(p))\times W(A_1)$ and p is odd. In this case the maps to the subgroups generated by $\{s,u\}$ and $\{t,u\}$ are non-zero. $\square$

We now consider the homology at $E^1_{2,0}$, using our calculations of the transfer maps.

Proposition 2.4.10. *The homology at $E^1_{2,0}$:*

$$\bigoplus_{t\in\mathcal{S}} H_0(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T).$$

is given by

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus\mathbb{Z}_2[E(\mathcal{D}_{even})]\oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2)$$

where the diagrams are as defined in Definition 2.1.4 and are viewed as 1-dimensional complexes.

Proof. Considering the calculations of the transfer maps in Lemmas 2.4.7 and 2.4.9 a splitting is observed. This is outlined in the diagram below.

$$\begin{tikzcd}
\bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \arrow[r, "d^1"] \arrow[d, two heads] & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) \arrow[r, "d^1"] \arrow[d, equal] & \bigoplus\limits_{t\in\mathcal{S}} H_0(W_t;\mathbb{Z}_t) \arrow[dddd, equal] \\
\bigoplus\limits_{W_T \text{ type } X} H_0(W_T;\mathbb{Z}_T) \arrow[r, "d^1"] & \bigoplus\limits_{\substack{T=\{s,t\}\\m(s,t)=2}} H_0(W_T;\mathbb{Z}_T) & \\
& \bigoplus & \\
& \bigoplus\limits_{\substack{T=\{s,t\}\\m(s,t)\neq 2 \text{ even}}} H_0(W_T;\mathbb{Z}_T) \\
& \bigoplus & \\
& \bigoplus\limits_{\substack{T=\{s,t\}\\m(s,t) \text{ odd}}} H_0(W_T;\mathbb{Z}_T) \arrow[r, "d^1"] & \bigoplus\limits_{t\in\mathcal{S}} H_0(W_T;\mathbb{Z}_T)
\end{tikzcd}$$

and calculating the homology of the top row in turn gives a splitting

$$\begin{gathered}
\operatorname{coker}\Big(\bigoplus_{W_T \text{ type } X} H_0(W_T,\mathbb{Z}_T) \xrightarrow{d^1} \bigoplus_{\substack{T=\{s,t\}\\m(s,t)=2}} H_0(W_T;\mathbb{Z}_T)\Big) \\
\bigoplus \\
\bigoplus_{\substack{T=\{s,t\}\\m(s,t)\neq 2 \text{ even}}} H_0(W_T;\mathbb{Z}_T) \\
\bigoplus \\
\ker\Big(\bigoplus_{\substack{T=\{s,t\}\\m(s,t) \text{ odd}}} H_0(W_T;\mathbb{Z}_T) \xrightarrow{d^1} \bigoplus_{t\in\mathcal{S}} H_0(W_T;\mathbb{Z}_T)\Big).
\end{gathered}$$

We now define an isomorphism $\epsilon = \epsilon_1 \oplus \epsilon_2 \oplus \epsilon_3$ from these three groups, to the three groups in the statement of the proposition:

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2).$$

We do this here in heavy detail, as this splitting technique is used regularly within the results of this chapter. For $\mathbb{Z}_2[E(\mathcal{D}_{even})]$, let $\{s,t\}$ be the basis element corresponding to the edge between s and t, and note that edges only exist if $m(s,t)$ is even and greater than 2. Recall that we denote the generator for $H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2$ by 1_T. Then ϵ_2 is defined by

$$\begin{array}{rrcl}\epsilon_2: & \bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\neq 2,\text{even}}} H_0(W_T;\mathbb{Z}_T) & \to & \mathbb{Z}_2[E(\mathcal{D}_{even})] \\ & 1_{\{s,t\}} & \mapsto & \{s,t\}\end{array}$$

and we note here that ϵ_2 is an isomorphism on inspection.

For $H_1(\mathcal{D}_{odd};\mathbb{Z}_2)$, note that when viewed as a simplicial complex, $\mathcal{D}_{odd}$ has no 2-cells, so $H_1(\mathcal{D}_{odd};\mathbb{Z}_2)=\ker(d:C_1\to C_0)$ for the simplicial differential d. Here C_1 is generated by edges $\{s,t\}$ between vertices s and t where $m(s,t)$ is odd, i.e. $C_1=\mathbb{Z}_2[E(\mathcal{D}_{odd})]$ and C_0 is generated by the vertices of $\mathcal{D}_{odd}$, given by the generating set S of W, i.e. $C_0=\mathbb{Z}_2[S]$. Moreover $d(\{s,t\})=s+t$. Recall from Lemma 2.4.7 that the transfer map is given on summands $H_0(W_{\{s,t\}};\mathbb{Z}_T)$ by $d^1(1_{\{s,t\}})=1_s+1_t$ if $m(s,t)$ is odd. Therefore we can define a chain map:

$$\begin{array}{rcl}\bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\text{ odd}}} H_0(W_T;\mathbb{Z}_T) & \to & \mathbb{Z}_2[E(\mathcal{D}_{odd})] \\ 1_{\{s,t\}} & \mapsto & \{s,t\}\end{array}$$

and this map induces an isomorphism between homologies, ϵ_3:

$$\begin{array}{rcl}\epsilon_3:\ker\Big(\bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\text{odd}}} H_0(W_T;\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{t\in\mathcal{S}}H_0(W_T;\mathbb{Z}_T)\Big) & \to & H_1(\mathcal{D}_{odd};\mathbb{Z}_2) \\ \ker\Big(\bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\text{odd}}} H_0(W_T;\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{t\in\mathcal{S}}H_0(W_T;\mathbb{Z}_T)\Big) & \to & \ker(d:\mathbb{Z}_2[E(\mathcal{D}_{odd})]\to\mathbb{Z}_2[S]).\end{array}$$

The map between the first groups is as follows:

$$\begin{array}{rcl}\epsilon_1:\operatorname{coker}\Big(\bigoplus_{W_T\text{ type }X} H_0(W_T,\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{\substack{T=\{s,t\}\\ m(s,t)=2}} H_0(W_T;\mathbb{Z}_T)\Big) & \to & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \\ 1_{\{s,t\}} & \mapsto & [\{s,t\}],\end{array}$$

where $[\{s,t\}]$ is the generator for the summand of $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ corresponding to the connected component containing $\{s,t\}$ (i.e. the component whose vertices are labelled by $\{s',t'\}$ with $\{s',t'\}$ equivalent under the relation $\approx$ to $\{s,t\}$).

Recall from Lemma 2.4.9 that the transfer map is given on summands $H_0(W_{\{s,t,u\}};\mathbb{Z}_T)$ by $d^1(1_{\{s,t,u\}})=1_{\{s,u\}}+1_{\{t,u\}}$ if W_T is of type X. Therefore generators of H_0 for triples of type X get mapped to sums of generators of H_0 corresponding to commuting pairs (elements of $S_{\bullet\bullet}$) which are equivalent to each other under $\sim$, i.e. they are in the same component of $\mathcal{D}_{\bullet\bullet}$. Therefore the map ϵ_1 is well defined and moreover it is an isomorphism. This concludes the proof. $\square$

2.4.11. Proof of Theorem A.

THEOREM 2.4.12. *Given a finite rank Coxeter group W with diagram $\mathcal{D}_W$, recall from Definition 2.1.4 the definition of the diagrams $\mathcal{D}_{\bullet\bullet}$, $\mathcal{D}_{odd}$ and $\mathcal{D}_{even}$. Then there is a natural isomorphism*

$$H_2(W;\mathbb{Z}) = H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2)$$

where in the first and final term of the right-hand-side the diagrams are considered as simplicial complexes consisting of 0-simplices (vertices of the diagram) and 1-simplices (edges of the diagram).

PROOF. The red diagonal of the isotropy spectral sequence in Figure 1 gives filtration quotients of $H_2(W;\mathbb{Z})$ on the E^∞ page. The E^2 page is as follows:

$$\begin{array}{c|cccc}
3 & ? & \vdots & & \\
2 & 0 & ? & \vdots & \\
1 & 0 & 0 & ? & \vdots \\
0 & ? & ? & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{even})] \oplus H_1(\mathcal{D}_{odd};\mathbb{Z}_2) & \cdots \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

Here all differentials d^r for $r \geq 2$ with source or target the $E_{2,0}$ position either originate at, or map to a zero group. Therefore the red diagonal on the limiting E^∞ page is given by the diagonal on the E^2 page. Since there is only one non zero group on the diagonal, there are no extension problems and this group gives $H_2(W;\mathbb{Z})$ as required. ☐

2.5. Calculation for $H_3(W;\mathbb{Z})$

The isotropy spectral sequence for the Coxeter group W has E^1 page the following

$$
\begin{array}{c|ccccccc}
\vdots & \vdots & & \vdots & & \vdots & & \vdots \quad \vdots \\
3 & H_3(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in S} H_3(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_3(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_3(W_T;\mathbb{Z}_T) \quad \cdots \\
2 & H_2(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in S} H_2(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_2(W_T;\mathbb{Z}_T) \quad \cdots \\
1 & H_1(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) \quad \cdots \\
0 & H_0(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in S} H_0(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \quad \cdots \\
\hline
 & 0 & & 1 & & 2 & & 3 \quad 4
\end{array}
$$

and the E^∞ page gives us $H_3(W;\mathbb{Z})$ (up to extension) on the blue diagonal.

2.5.1. Free resolution for Coxeter groups. In this section we follow the paper *Cohomology of Coxeter and Artin groups* by De Concini and Salvetti [**18**]. They describe a free resolution of $\mathbb{Z}$ over $\mathbb{Z}W$ for a finite Coxeter group W with generating set S. We will use this throughout this section to calculate the low dimensional homologies of finite Coxeter groups that appear as summands in the entries of the spectral sequence.

The free resolution is denoted (C_*, δ_*) and defined as follows: C_k is a free $\mathbb{Z}W$ module with basis elements $e(\Gamma)$ for Γ a flag of subsets of S with cardinality k, that is Γ in S_k where:

$$S_k := \{\Gamma = (\Gamma_1 \supset \Gamma_2 \supset \cdots) \mid \Gamma_1 \subset S, \sum_{i\geq 1} |\Gamma_i| = k\}.$$

The differential is defined using minimal left coset representatives of parabolic subgroups. For τ in Γ_i, let $W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}$ be the set of minimal left coset representatives of $W_{\Gamma_i\setminus\{\tau\}}$ in W_{Γ_i}. Then $\delta_k : C_k \to C_{k-1}$ is $\mathbb{Z}W$ linear and defined as follows

$$\delta_k e(\Gamma) = \sum_{\substack{i\geq 1\\ |\Gamma_i|>|\Gamma_{i+1}|}} \sum_{\tau\in\Gamma_i} \sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma') \tag{4}$$

where the flag Γ' in C_{k-1} is given by

$$\Gamma' := (\Gamma_1 \supset \cdots \supset \Gamma_{i-1} \supset (\Gamma_i\setminus\{\tau\}) \supset \beta^{-1}\Gamma_{i+1}\beta \supset \beta^{-1}\Gamma_{i+2}\beta \supset \cdots)$$

and the exponent $\alpha(\Gamma, i, \tau, \beta)$ is given by a formula in terms of Γ, i, τ and β which we define below. This is well defined from Lemma 1.2.6. We choose an ordering for the set of generators S and let $\sigma(\beta, \Gamma_k)$ be the number of inversions, with respect to this ordering, in the map $\Gamma_k \to \beta^{-1}\Gamma_k\beta$. We let $\mu(\Gamma_i, \tau)$ be the number of generators in Γ_i which are less than or equal to τ in the ordering on S. Given this, the exponent is described by the following formula:

$$\alpha(\Gamma, i, \tau, \beta) = i \cdot \ell(\beta) + \sum_{k=1}^{i-1} |\Gamma_k| + \mu(\Gamma_i, \tau) + \sum_{k=i+1}^{d} \sigma(\beta, \Gamma_k).$$

During this proof we adopt the convention that the generators are always ordered alphabetically (e.g. $s < t < u$). We also denote the generator corresponding to a flag of length d, $(\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_d)$ by $\Gamma_{\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_d}$, where we omit the set notation for each Γ_i. For example Γ_s, $\Gamma_{s\supset s}$, or $\Gamma_{s,t\supset s}$ (which corresponds to $\Gamma = \{s,t\} \supset \{s\}$).

Lemma 2.5.2. *In all computations of the differential δ_p for $0 \leq p \leq 4$,*

$$\sum_{k=i+1}^{d} \sigma(\beta, \Gamma_k) = 0.$$

Proof. The differential $\delta_p : C_p \to C_{p-1}$ is nonzero when for some $i \geq 0$ we have $|\Gamma_i| > |\Gamma_{i+1}|$, and the sum

$$\sum_{k=i+1}^{d} \sigma(\beta, \Gamma_k)$$

is over k where k starts at $i+1$ and ends at d, for the flag $\Gamma_{\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_d}$. Therefore there are no terms in this sum unless Γ_{i+1} is non-empty. Let s, t, u be in the generating set S. Generators in C_0 have the form $\Gamma_\emptyset$, in C_1 have the form Γ_s, and in C_2 have the form $\Gamma_{s\supset s}$ or Γ_{st}. Therefore none of these generators satisfy $|\Gamma_i| > |\Gamma_{i+1}|$ for Γ_{i+1} non-empty. The only generators in C_3 and C_4 which satisfy the property are $\Gamma_{st\supset s}$ in C_3 or $\Gamma_{st\supset s\supset s}$ and $\Gamma_{stu\supset s}$ in C_4. For all of these generators, the property is satisfied for Γ_{i+1} a singleton. Since $\sigma(\beta, \Gamma_k)$ calculates the number of inversions in the map $\Gamma_k \to \beta^{-1}\Gamma_k\beta$ from this singleton to another set, the number of inversions will be zero since an inversion can only take place when there are two or more elements in the source set. This completes the proof. $\square$

We therefore omit the $\sigma(\beta, \Gamma_k)$ term from our calculations in this chapter, as we only ever calculate differentials δ_p for $0 \leq p \leq 4$.

Example 2.5.3. We give an example of the resolution for finite Coxeter groups with one generator $S = \{s\}$, from C_3 to C_0.

$$C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

Generators:

$$\Gamma_{s\supset s\supset s} \qquad \Gamma_{s\supset s} \qquad \Gamma_s \qquad \Gamma_\emptyset$$

Differentials:

$$\Gamma_s \longmapsto (s-1)\Gamma_\emptyset$$

$$\Gamma_{s\supset s} \longmapsto (1+s)\Gamma_s$$

$$\Gamma_{s\supset s\supset s} \longmapsto (s-1)\Gamma_{s\supset s}$$

The differential from Γ_s to $\Gamma_\emptyset$ is given by the following formula, noting that coset representatives of $W_\emptyset$ in W_s are e and s. We recall the formula for $\delta_k(e(\Gamma))$ from Equation (4).

$$\begin{aligned}
\delta_1(\Gamma_s) &= \sum_{i=1}\sum_{s}\sum_{\beta=e,s}(-1)^{\alpha(\Gamma_s,1,s,\beta)}\beta\Gamma_\emptyset \\
&= \sum_{\beta=e,s}(-1)^{\alpha(\Gamma_s,1,s,\beta)}\beta\Gamma_\emptyset \\
&= (-1)^1 e\Gamma_\emptyset + (-1)^2 s\Gamma_\emptyset \\
&= (s-1)\Gamma_\emptyset
\end{aligned}$$

where we compute

$$\begin{aligned}
\alpha(\Gamma_s,1,s,e) &= 1\ell(e) + \sum_{k=1}^{0}|\Gamma_k| + \mu(s,s) \\
&= 0+0+1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\alpha(\Gamma_s,1,s,s) &= 1\ell(s) + \sum_{k=1}^{0}|\Gamma_k| + \mu(s,s) \\
&= 1+0+1 \\
&= 2.
\end{aligned}$$

Similarly the differential $\delta_2 : C_2 \to C_1$ is given by

$$\begin{aligned}
\delta_2(\Gamma_{s\supset s}) &= \sum_{i=2}\sum_{s}\sum_{\beta=e,s}(-1)^{\alpha(\Gamma_{s\supset s},2,s,\beta)}\beta\Gamma_s \\
&= \sum_{\beta=e,s}(-1)^{\alpha(\Gamma_{s\supset s},2,s,\beta)}\beta\Gamma_s \\
&= (-1)^2 e\Gamma_s + (-1)^4 s\Gamma_s \\
&= (1+s)\Gamma_s
\end{aligned}$$

where we compute

$$\begin{aligned}
\alpha(\Gamma_{s\supset s},2,s,e) &= 2\ell(e) + \sum_{k=1}^{1}|\Gamma_k| + \mu(s,s) \\
&= 0+1+1 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\alpha(\Gamma_{s\supset s},2,s,s) &= 2\ell(s) + \sum_{k=1}^{1}|\Gamma_k| + \mu(s,s) \\
&= 2+1+1 \\
&= 4.
\end{aligned}$$

Finally, the differential $\delta_3 : C_3 \to C_2$ is given by

$$\begin{aligned}
\delta_3(\Gamma_{s\supset s\supset s}) &= \sum_{i=3}\sum_{s}\sum_{\beta=e,s}(-1)^{\alpha(\Gamma_{s\supset s\supset s},3,s,\beta)}\beta\Gamma_{s\supset s} \\
&= \sum_{\beta=e,s}(-1)^{\alpha(\Gamma_{s\supset s\supset s},3,s,\beta)}\beta\Gamma_{s\supset s} \\
&= (-1)^3 e\Gamma_{s\supset s} + (-1)^6 s\Gamma_{s\supset s} \\
&= (s-1)\Gamma_{s\supset s}
\end{aligned}$$

where we compute

$$\begin{aligned}
\alpha(\Gamma_{s\supset s\supset s},3,s,e) &= 3\ell(e) + \sum_{k=1}^{2}|\Gamma_k| + \mu(s,s) \\
&= 0+2+1 \\
&= 3
\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s\supset s\supset s}, 3, s, s) &= 3\ell(s) + \sum_{k=1}^{2} |\Gamma_k| + \mu(s,s)\\ &= 3+2+1\\ &= 6.\end{aligned}$$

DEFINITION 2.5.4. Define $p(s,t;j)$ to be the alternating product of s and t of length j, *ending* in an s (as opposed to $\pi(s,t;j)$ which is the alternating product *starting* in an s) i.e.

$$p(s,t;j) = \overbrace{\ldots sts}^{\text{length j}}$$

EXAMPLE 2.5.5. We give an example of the resolution for finite Coxeter groups with two generators $S = \{s,t\}$, from C_3 to C_0 and with $m(s,t)$ finite. Here, the resolution is given on a landscape page for ease of reading, and the calculations of the differentials are given in Appendix B.

$$C_3 \xrightarrow{\ \delta_3\ } C_2 \xrightarrow{\ \delta_2\ } C_1 \xrightarrow{\ \delta_1\ } C_0$$

Generators:

$$\begin{array}{llll}
\Gamma_{s\supset s\supset s} & \Gamma_{s\supset s} & \Gamma_s & \Gamma_\emptyset \\
\Gamma_{t\supset t\supset t} & \Gamma_{t\supset t} & \Gamma_t & \\
\Gamma_{s,t\supset s} & \Gamma_{s,t} & & \\
\Gamma_{s,t\supset t} & & &
\end{array}$$

Differentials:

$$\Gamma_s \longmapsto (s-1)\Gamma_\emptyset$$

$$\Gamma_t \longmapsto (t-1)\Gamma_\emptyset$$

$$\Gamma_{s\supset s} \longmapsto (1+s)\Gamma_s$$

$$\Gamma_{t\supset t} \longmapsto (1+t)\Gamma_t$$

$$\Gamma_{s,t} \longmapsto \sum_{j=0}^{m(s,t)-1} (-1)^{j+1} p(s,t;j)\Gamma_t + \sum_{g=0}^{m(s,t)-1} (-1)^{g+2} p(t,s;g)\Gamma_s$$

$$\Gamma_{s\supset s\supset s} \longmapsto (s-1)\Gamma_{s\supset s}$$

$$\Gamma_{t\supset t\supset t} \longmapsto (t-1)\Gamma_{t\supset t}$$

$$\Gamma_{s,t\supset s} \mapsto \begin{array}{ll} (1-p(t,s;m(s,t)-1))\Gamma_{s\supset s} - (1+s)\Gamma_{st} & \text{if } m(s,t) \text{ even} \\ \Gamma_{s\supset s} - p(s,t;m(s,t)-1)\Gamma_{t\supset t} - (1+s)\Gamma_{st} & \text{if } m(s,t) \text{ odd} \end{array}$$

$$\Gamma_{s,t\supset t} \mapsto \begin{array}{ll} (-1+p(s,t;m(s,t)-1))\Gamma_{t\supset t} - (1+t)\Gamma_{st} & \text{if } m(s,t) \text{ even} \\ -\Gamma_{t\supset t} + p(t,s;m(s,t)-1)\Gamma_{s\supset s} - (1+t)\Gamma_{st} & \text{if } m(s,t) \text{ odd} \end{array}$$

The entries in the spectral sequence which we wish to compute are in fact homologies of finite Coxeter groups with twisted coefficients $\mathbb{Z}_T$ given a generating set T, in which the action of the generators on $\mathbb{Z}_T$ is given by negation. To calculate the twisted homologies we tensor the resolution with $\mathbb{Z}$ under the group action. We show this in the case of our two examples.

Example 2.5.6. We give an example of the tensored resolution for finite Coxeter groups with one generator $S = \{s\}$, from C_3 to C_0. We consider the resolution of Example 2.5.3 and upon tensoring with $\mathbb{Z}$ under the group action, group elements act as negation if they have odd length and the identity if they have even length. This gives the following resolution:

$$\mathbb{Z} \underset{W_s}{\otimes} C_3 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_s}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_s}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_s}{\otimes} C_0$$

Generators:

$$1 \otimes \Gamma_{s\supset s\supset s} \qquad 1 \otimes \Gamma_{s\supset s} \qquad 1 \otimes \Gamma_s \qquad 1 \otimes \Gamma_\emptyset$$

Differentials:

$$1 \otimes \Gamma_s \longmapsto \begin{array}{l} 1 \otimes ((s-1)\Gamma_\emptyset) \\ = -2(1 \otimes \Gamma_\emptyset) \end{array}$$

$$1 \otimes \Gamma_{s\supset s} \longmapsto \begin{array}{l} 1 \otimes ((1+s)\Gamma_s) \\ = 0 \end{array}$$

$$1 \otimes \Gamma_{s\supset s\supset s} \longmapsto \begin{array}{l} 1 \otimes ((s-1)\Gamma_{s\supset s}) \\ = -2(1 \otimes \Gamma_{s\supset s}) \end{array}$$

Example 2.5.7. We give an example of the tensored resolution for finite Coxeter groups with two generators $T = \{s, t\}$, from C_3 to C_0 and with $m(s,t)$ finite. We consider the resolution of Example 2.5.5 and upon tensoring with $\mathbb{Z}$ under the group action, this gives the following resolution:

$$\mathbb{Z} \underset{W_T}{\otimes} C_3 \xrightarrow{\delta_3} \mathbb{Z} \underset{W_T}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_T}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_T}{\otimes} C_0$$

Generators:

$1 \otimes \Gamma_{s\supset s\supset s}$	$1 \otimes \Gamma_{s\supset s}$	$1 \otimes \Gamma_s$	$1 \otimes \Gamma_\emptyset$
$1 \otimes \Gamma_{t\supset t\supset t}$	$1 \otimes \Gamma_{t\supset t}$	$1 \otimes \Gamma_t$	
$1 \otimes \Gamma_{s,t\supset s}$	$1 \otimes \Gamma_{st}$		
$1 \otimes \Gamma_{s,t\supset t}$			

Differentials:

$$1 \otimes \Gamma_s \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_t \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_{s\supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{t\supset t} \longmapsto 0$$

$$1 \otimes \Gamma_{s,t} \longmapsto \begin{array}{l} 1 \otimes (\sum_{j=0}^{m(s,t)-1} (-1)^{j+1} p(s,t;j)\Gamma_t \\ + \sum_{g=0}^{m(s,t)-1} (-1)^{g+2} p(t,s;g)\Gamma_s) \\ = -m(s,t)(1 \otimes \Gamma_t) + m(s,t)(1 \otimes \Gamma_s) \end{array}$$

$$1 \otimes \Gamma_{s\supset s\supset s} \longmapsto \begin{array}{l} 1 \otimes (s-1)\Gamma_{s\supset s} \\ = -2(1 \otimes \Gamma_{s\supset s}) \end{array}$$

$$1 \otimes \Gamma_{t\supset t\supset t} \longmapsto \begin{array}{l} 1 \otimes (t-1)\Gamma_{t\supset t} \\ = -2(1 \otimes \Gamma_{t\supset t}) \end{array}$$

$$1 \otimes \Gamma_{s,t\supset s} \longmapsto \begin{array}{ll} 2(1 \otimes \Gamma_{s\supset s}) & \text{if } m(s,t) \text{ even} \\ 1 \otimes \Gamma_{s\supset s} - 1 \otimes \Gamma_{t\supset t} & \text{if } m(s,t) \text{ odd} \end{array}$$

$$1 \otimes \Gamma_{s,t\supset t} \longmapsto \begin{array}{ll} -2(1 \otimes \Gamma_{t\supset t}) & \text{if } m(s,t) \text{ even} \\ -1 \otimes \Gamma_{t\supset t} + 1 \otimes \Gamma_{s\supset s} & \text{if } m(s,t) \text{ odd} \end{array}$$

2.5.8. Collapse map. In this section we define a chain map, which we call the *collapse map*, between De Concini and Salvetti's resolution for a finite Coxeter group W, and for a subgroup W_T [**18**].

In the isotropy spectral sequence for the Davis complex, introduced in Section 2.3.14, we calculate that on the E^1 page, the d^1 differential has the form of a transfer map between summands $H_*(W_T;\mathbb{Z}_T)$ and $H_*(W_U;\mathbb{Z}_U)$ for $U \subset T$, given in Proposition 2.3.15. In the following sections we calculate these twisted homology groups using the De Concini and Salvetti resolution. Upon applying the transfer map to a generator of the homology $H_*(W_T;\mathbb{Z}_T)$, the image will be in terms of the resolution for the group W_T. However we would like the image to be in terms of the resolution for W_U and so we then apply the collapse map in the appropriate degree to achieve this.

We first recall the following Lemmas, which are re-workings of Lemmas from [**27**], into settings relevant to this section. Recall from Definition 1.1.6 that $\pi(a,b;k)$ is defined to be the word of length k, given by the alternating product of a and b i.e.

$$\pi(a,b;k) = \overbrace{abab\ldots}^{\text{length k}}$$

Lemma 2.5.9 (Deodhar's Lemma, see Geck and Pfeiffer [**27**, 2.1.2]). *Let W_T be a spherical subgroup of a finite Coxeter group W. Let v be $(T,\emptyset)$-reduced (as defined in Definition 1.2.7) and let s be in S, the generating set for W. Then either vs is $(T,\emptyset)$-reduced or $vs = tv$ for some t in T.*

Lemma 2.5.10 (see Geck and Pfeiffer [**27**, 1.2.1]). *If s,u are in S, $m(s,u)$ is finite, and w in W satisfies $\ell(ws) < \ell(w)$ and $\ell(wu) < \ell(w)$ then $w = w'(\pi(s,u;m(s,u)))$ where w' is $(\emptyset, W_{\{s,u\}})$-reduced, as defined in Definition 1.2.7.*

Definition 2.5.11. Denote the De Concini - Salvetti resolution for W by (C_*,δ_*) and for W_T by (D_*,δ_*). We define the *collapse map in degree i* to be the W_T-equivariant linear map $f_i : C_i \to D_i$ for $0 \le i \le 2$ as shown below.

$$\begin{array}{ccccccccc}
\xrightarrow{\delta_3} & C_2 & \xrightarrow{\delta_2} & C_1 & \xrightarrow{\delta_1} & C_0 & \xrightarrow{\delta_0} & \mathbb{Z} \\
 & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| \\
\xrightarrow{\delta_3} & D_2 & \xrightarrow{\delta_2} & D_1 & \xrightarrow{\delta_1} & D_0 & \xrightarrow{\delta_0} & \mathbb{Z}.
\end{array}$$

As a $\mathbb{Z}[W]$ module, C_* has basis given by $e(\Gamma)$, so as a $\mathbb{Z}[W_T]$ module, C_* has basis given by $v \cdot e(\Gamma)$, for v a $(T,\emptyset)$-reduced element of W. We therefore define f_i on generators of C_i multiplied on the left by v and extend the map linearly and W_T-equivariantly. By Deodhar's lemma (Lemma 2.5.9) for $s \in S$, vs is either $(T,\emptyset)$-reduced or $vs = tv$ for some t in T. This gives us the cases in each definition.

$$f_0(v\Gamma_\emptyset) = \Gamma_\emptyset,$$

$$f_1(v\Gamma_s) = \begin{cases} 0 & vs \text{ is } (T,\emptyset) \text{ reduced} \\ \Gamma_t & vs = tv \text{ for } t \in T \end{cases}$$

$$f_2(v\Gamma_{s\supset s}) \begin{cases} 0 & vs \text{ is } (T,\emptyset) \text{ reduced} \\ \Gamma_{t\supset t} & vs = tv \text{ for } t \in T \end{cases}$$

$$f_2(v\Gamma_{su}) = \begin{cases} \Gamma_{tr} & vs = tv \text{ and } vu = rv \text{ for } t, r \in T \\ 0 & \text{otherwise.} \end{cases}$$

The remainder of this section is devoted to proving that f_* is a chain map.

LEMMA 2.5.12. *The following square commutes:*

$$\begin{tikzcd} C_0 \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\ D_0 \arrow[r, "\delta_0"] & \mathbb{Z}. \end{tikzcd}$$

PROOF. On $w\Gamma_\emptyset$ for w in W, the square is given by

$$\begin{tikzcd} w\Gamma_\emptyset \arrow[r, "\delta_0"] \arrow[d, mapsto, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\ f_0(w\Gamma_\emptyset) \arrow[r, "\delta_0"] & \mathbb{Z} \end{tikzcd}$$

since f_0 is defined W_T-equivariantly then if $w = tv$ for t in W_T and v a $(T,\emptyset)$-reduced element then from Definition 2.5.11

$$f_0(w\Gamma_\emptyset) = f_0(tv\Gamma_\emptyset) = t \cdot f_0(v\Gamma_\emptyset) = t\Gamma_\emptyset.$$

It follows since δ_0 maps all generators to 1 that the square commutes. □

LEMMA 2.5.13. *The following square commutes*

$$\begin{tikzcd} C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\ D_1 \arrow[r, "\delta_1"] & D_0. \end{tikzcd}$$

PROOF. Since all maps are W_T-equivariant, let $w = tv$ for t in W_T and v a $(T,\emptyset)$-reduced element. Then we need only consider the square on generators multiplied by v. We recall the image of δ_1 from Example 2.5.3.

$$\begin{tikzcd} v\Gamma_s \arrow[r, "\delta_1"] \arrow[d, mapsto, "f_1"'] & v(s-1)\Gamma_\emptyset \arrow[d, mapsto, "f_0"'] \\ f_1(v\Gamma_s) \arrow[r, "\delta_1"] & f_0(v(s-1)\Gamma_\emptyset). \end{tikzcd}$$

Here there are two cases for the element vs, given by Lemma 2.5.9 which give the following cases for f_0, from Definition 2.5.11:

$$f_0(v(s-1)\Gamma_\emptyset) = \begin{cases} 0 & vs\,(T,\emptyset) \text{ reduced} \\ (t-1)\Gamma_\emptyset & vs = tv. \end{cases}$$

This is precisely the image of $f_1(v\Gamma_s)$ from Definition 2.5.11, under the differential δ_1. Therefore the square commutes. $\square$

Lemma 2.5.14. *For s and u in S, consider the following three cases, given by Deodhar's Lemma 2.5.9:*

(1) Neither vs or vu are $(T,\emptyset)$-reduced, that is $vs = tv$ and $vu = rv$ for t and r in T.
(2) One of vs and vu is $(T,\emptyset)$-reduced, without loss of generality let $vs = tv$ and vu is $(T,\emptyset)$-reduced.
(3) Both vs and vu are $(T,\emptyset)$-reduced.

Recall the definition of $p(s,t;m)$ from Definition 2.5.4. Then

$$f_1\left(v\big(\textstyle\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u + \sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\big)\right)$$
$$= \begin{cases} \delta_2(\Gamma_{tr}) & \text{in Case } (1) \\ 0 & \text{in Case } (2) \\ 0 & \text{in Case } (3). \end{cases}$$

Proof. We prove the lemma case by case. For Case (1), since f_1 acts W_T-equivariantly,

$$f_1(v(p(s,u;j)\Gamma_u)) = f_1(p(t,r;j)v\Gamma_u) = p(t,r;j)(f_1(v\Gamma_u)) = p(t,r;j)\Gamma_r$$

and similarly

$$f_1(vp(u,s;g)\Gamma_s) = p(r,t;g)\Gamma_t.$$

Furthermore, $m(t,r) = m(s,u)$ since

$$\pi(t,r;m(s,u))v = v\pi(s,u;m(s,u)) = v\pi(u,s;m(s,u)) = \pi(r,t;m(s,u))v,$$

and right multiplication by v^{-1} gives that $\pi(t,r;m(s,u)) = \pi(r,t;m(s,u))$, so $m(t,r)$ is a divisor of $m(s,u)$. Furthermore, applying a similar argument in reverse gives $m(s,u)$ is a divisor of $m(t,r)$, and so $m(s,u) = m(t,r)$.

Therefore since f_1 acts linearly

$$\begin{aligned}
& f_1\left(v\big(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\big)\right)\\
= & \sum_{j=0}^{m(t,r)-1}(-1)^{j+1}p(t,r;j)\Gamma_r+\sum_{g=0}^{m(t,r)-1}(-1)^{g+2}p(r,t;g)\Gamma_t\\
= & \delta_2(\Gamma_{tr})
\end{aligned}$$

in the setting of Case (1).

For Case (2), we first prove that if $vs = tv$ and vu is $(T,\emptyset)$-reduced, then $v(\pi(u,s;k))$ is also $(T,\emptyset)$-reduced for all $2 \le k \le m(s,u)-1$. First we note that since $vs = tv$, from Lemma 2.5.9 $\ell(vs) > \ell(v)$. Suppose $v(\pi(u,s;k))$ was not $(T,\emptyset)$-reduced and choose minimal k for which this is the case (so $v(\pi(u,s;k-1))$ is $(T,\emptyset)$-reduced). Then for some q in T it follows $v(\pi(u,s;k)) = qv(\pi(u,s;k-1))$ and so $w = v(\pi(u,s;k))$ satisfies the hypothesis of Lemma 2.5.10, that is $\ell(wu) < \ell(w)$ and $\ell(ws) < \ell(w)$. Therefore $w = w'\pi(u,s;m(s,u))) = v(\pi(u,s;k))$, so by right multiplication by $(\pi(u,s;k))^{-1}$ we have $v = w'p(s,u;m(s,u)-k)$, where we recall $p(s,u;m)$ is the alternating product of s and u of length m and ending in s. Therefore v satisfies $\ell(vs) < \ell(v)$. This contradicts $vs = tv$, so we must have $v(\pi(u,s;k))$ is also $(T,\emptyset)$-reduced for all $2 \le k \le m(s,u)-1$. Computing f_1 on the expressions of the sum therefore gives:

$$f_1(v(p(s,u;j)\Gamma_u)) = \begin{cases}
f_1(v(\pi(u,s;j)\Gamma_u)) = 0 & j \text{ is even, } j \neq m(s,u)-1\\
t\cdot f_1(v\pi(u,s;j-1)\Gamma_u) = t\cdot 0 = 0 & j \text{ is odd, } j \neq m(s,u)-1\\
f_1(v\pi(u,s;m(s,t)-1)\Gamma_u) = \Gamma_t & j = m(s,u)-1 \text{ and is even}\\
t\cdot f_1(v\pi(u,s;m(s,t)-2)\Gamma_u) = t\cdot 0 & j = m(s,u)-1 \text{ and is odd}
\end{cases}$$

and similarly

$$f_1(vp(u,s;g)\Gamma_s) = \begin{cases}
f_1(v\Gamma_s) = \Gamma_t & g = 0\\
t\cdot f_1(v\pi(u,s;g-1)\Gamma_s) = t\cdot 0 = 0 & g \text{ is even, } g \notin \{0, m(s,u)-1\}\\
f_1(v\pi(u,s;g)\Gamma_s) = 0 & g \text{ is odd, } g \neq m(s,u)-1\\
t\cdot f_1(v\pi(u,s;m(s,t)-2)\Gamma_s) = t\cdot 0 = 0 & g = m(s,u)-1 \text{ and is even}\\
f_1(v\pi(u,s;m(s,t)-1)\Gamma_s) = \Gamma_t & g = m(s,u)-1 \text{ and is odd}
\end{cases}$$

so it follows

$$f_1\left(v\Big(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\Big)\right)$$
$$=\begin{cases}\Gamma_t+(-1)^{m(s,t)-1+2}\Gamma_t=0 & \text{if } m(s,u) \text{ even}\\ \Gamma_t+(-1)^{m(s,u)-1+1}\Gamma_t=0 & \text{if } m(s,u) \text{ odd}\end{cases}$$

in the setting of Case (2).

For Case (3), if both vs and vu are $(T,\emptyset)$-reduced, by the same argument as in Case (2), $v(\pi(u,s;k))$ and $v(\pi(s,u;k))$ is also $(T,\emptyset)$-reduced for all $2\le k\le m(s,u)$. It follows that computing f_1 gives:

$$f_1\left(v\Big(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\Big)\right)=0$$

in the setting of Case (3). ☐

LEMMA 2.5.15. *The following square commutes*

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & C_1 \arrow[d, "f_1"'] \\ D_2 \arrow[r, "\delta_2"] & D_1. \end{tikzcd}$$

PROOF. Since all maps are W_T-equivariant, let $w=tv$ for t in W_T and v a $(T,\emptyset)$-reduced element. Then we need only consider the square on generators multiplied by v. We recall the image of δ_2 from Example 2.5.5. We must consider both forms of generators of C_2:

$$\begin{tikzcd} v\Gamma_{s\supset s} \arrow[r, "\delta_2"] \arrow[d, mapsto, "f_2"'] & v(1+s)\Gamma_s \arrow[d, mapsto, "f_1"'] \\ f_2(v\Gamma_{s\supset s}) \arrow[r, "\delta_2"] & f_1(v(1+s)\Gamma_s) \end{tikzcd}
\qquad
\begin{tikzcd} v\Gamma_{s,u} \arrow[r, "\delta_2"] \arrow[d, mapsto, "f_2"'] & v\big(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,t;j)\Gamma_u + \sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\big) \arrow[d, mapsto, "f_1"'] \\ f_2(v\Gamma_{s,u}) \arrow[r, "\delta_2"] & f_1\big(v\big(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_t + \sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\big)\big). \end{tikzcd}$$

Computing $f_1(v(1+s)\Gamma_s)$ we have

$$f_1(v(1+s)\Gamma_s)=\begin{cases}0 & vs \text{ is } (T,\emptyset) \text{ reduced}\\ (1+t)\Gamma_t & vs=tv.\end{cases}$$

This is precisely the image of $f_2(v\Gamma_{s\supset s})$ from Definition 2.5.11, under the differential δ_2. Therefore the left hand square commutes.

The bottom right entry of the right hand square is given in Lemma 2.5.14. This is precisely the image of $f_2(v\Gamma_{s,u})$ from Definition 2.5.11, under the differential δ_2. Therefore the left hand square commutes. □

Proposition 2.5.16. *The maps f_0, f_1 and f_2 in Definition 2.5.11 form part of a chain map $f_\bullet : C_\bullet \to D_\bullet$.*

Proof. This is a consequence of Lemmas 2.5.12, 2.5.13 and 2.5.15, which show that all squares in the following diagram commute

$$\begin{tikzcd}
{} \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & C_0 \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\
{} \arrow[r, "\delta_3"] & D_2 \arrow[r, "\delta_2"] & D_1 \arrow[r, "\delta_1"] & D_0 \arrow[r, "\delta_0"] & \mathbb{Z}.
\end{tikzcd}$$

□

2.5.17. Homology at $E^1_{0,3}$. Recall the isotropy spectral sequence for the Davis complex of a Coxeter group W has E^1 page as follows:

$$\begin{array}{c|ccccccccc}
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
3 & H_3(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in\mathcal{S}} H_3(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_3(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_3(W_T;\mathbb{Z}_T) & & \cdots \\
2 & H_2(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in\mathcal{S}} H_2(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_2(W_T;\mathbb{Z}_T) & & \cdots \\
1 & H_1(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in\mathcal{S}} H_1(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) & & \cdots \\
0 & H_0(W_\emptyset;\mathbb{Z}_\emptyset) & \xleftarrow{d^1} & \bigoplus_{t\in\mathcal{S}} H_0(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) & & \cdots \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4
\end{array}$$

and the E^∞ page gives us filtration quotients of $H_3(W;\mathbb{Z})$ on the blue diagonal.

Then the $E^1_{0,3}$ entry is zero because it is the third integral homology of the trivial group, $H_3(W_\emptyset;\mathbb{Z}_\emptyset) = 0$, on the E^1 page. Therefore $E^2_{0,3}$ and $E^\infty_{0,3}$ are zero.

2.5.18. Homology at $E^1_{1,2}$. To calculate this, we use the De Concini - Salvetti resolution [**18**] to compute the twisted homologies, and the transfer and collapse map to compute the differentials for the following section of the spectral sequence:

$$H_2(W_\emptyset;\mathbb{Z}_\emptyset) \xleftarrow{d^1} \bigoplus_{t\in S} H_2(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T).$$

We note that $H_2(W_\emptyset;\mathbb{Z}_\emptyset)=0$ since it is the second homology of the trivial group.

Lemma 2.5.19. *The second twisted homology for a one generator Coxeter group W_t is $H_2(W_t;\mathbb{Z}_t)=\mathbb{Z}_2$, generated by $1\otimes\Gamma_{s\supset s}$ in the De Concini - Salvetti resolution.*

Proof. This calculation is in Appendix B. □

Lemma 2.5.20. *If $T=\{s,t\}$ then the second twisted homology the following,*

$$H_2(W_T;\mathbb{Z}_T)=\begin{cases}\mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{if } m(s,t) \text{ is even}\\ \mathbb{Z}_2 & \text{if } m(s,t) \text{ is odd,}\end{cases}$$

and in the De Concini - Salvetti resolution this is generated by $1\otimes\Gamma_{s\supset s}$ and $1\otimes\Gamma_{t\supset t}$ when $m(s,t)$ is even, with these generators being identified when $m(s,t)$ is odd.

Proof. This calculation is in Appendix B. □

Lemma 2.5.21. *The transfer map*

$$d^1:\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T)\to\bigoplus_{t\in S} H_2(W_t;\mathbb{Z}_t)$$

restricted to the summand relating to $T=\{s,t\}$ in the source and restricted to the summand s and t in the image is given by

$$\begin{aligned} d^1_{T,s}: H_2(W_{\{s,t\}};\mathbb{Z}_T) &\to H_2(W_s;\mathbb{Z}_s)\\ 1\otimes\Gamma_{s\supset s}, 1\otimes\Gamma_{t\supset t} &\mapsto 0 \text{ if } m(s,t) \text{ even}\\ 1\otimes\Gamma_{s\supset s} &\mapsto 1\otimes\Gamma_{s\supset s} \text{ if } m(s,t) \text{ odd}\end{aligned}$$

$$\begin{aligned} d^1_{T,t}: H_2(W_{\{s,t\}};\mathbb{Z}_T) &\to H_2(W_t;\mathbb{Z}_t)\\ 1\otimes\Gamma_{s\supset s}, 1\otimes\Gamma_{t\supset t} &\mapsto 0 \text{ if } m(s,t) \text{ even}\\ 1\otimes\Gamma_{s\supset s} &\mapsto 1\otimes\Gamma_{t\supset t} \text{ if } m(s,t) \text{ odd.}\end{aligned}$$

Proof. This calculation is in Appendix B. □

Proposition 2.5.22. *The $E^2_{1,2}$ entry on the E^2 page of the spectral sequence is given by $H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$.*

PROOF. We have the following groups and differentials on the E^1 page:

$$H_2(W_\emptyset;\mathbb{Z}_\emptyset) \xleftarrow{d^1} \bigoplus_{t\in S} H_2(W_t;\mathbb{Z}_t) \xleftarrow{\quad d^1 \quad} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T)$$

$$0 \xleftarrow{\quad d^1 \quad} \bigoplus_{t\in S}\mathbb{Z}_2 \xleftarrow{\quad d^1 \quad} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} \begin{cases}\mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{if } m(s,t) \text{ even}\\ \mathbb{Z}_2 & \text{if } m(s,t) \text{ odd.}\end{cases}$$

The left hand map is the zero map and the right hand map is defined via Lemma 2.5.21. Applying the splitting technique as in the proof of the $H_2(W;\mathbb{Z})$ calculation (i.e. as in Proposition 2.4.10), we can equate the kernel of the left hand map over the image of the right hand map to the 0th homology with $\mathbb{Z}_2$ coefficients of the diagram with only odd edges, $\mathcal{D}_{odd}$. □

2.5.23. Homology at $E^1_{2,1}$. We use The De-Concini Salvetti resolution to calculate

$$E^1_{3,1} = \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T),$$

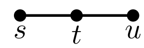
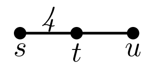
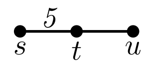
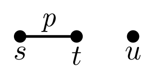
the first twisted homology of spherical subgroups with 3 generators. After calculating these, we use the transfer and collapse map to compute the d^1 differentials and therefore we can compute $E^2_{2,1}$. The E^1 page as $E^1_{2,1}$ has the following form:

$$\bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T).$$

Using De Concini - Salvetti we can calculate the first homology of the spherical subgroups. The formulation of the twisted resolutions and homology calculations are of a similar nature to those for the 1 generator and 2 generator cases that we have described in some detail throughout the preceding sections.

PROPOSITION 2.5.24. *The first homology $H_1(W_T;\mathbb{Z}_T)$ is as follows for spherical subgroups W_T with $T=\{s,t,u\}$. Generators are given by the De Concini - Salvetti resolution for W_T: we let*

$$\alpha = (1\otimes\Gamma_s)-(1\otimes\Gamma_t) \textit{ and } \beta=(1\otimes\Gamma_s)-(1\otimes\Gamma_u).$$

W_T	$\mathcal{D}_{W_T}$	$H_1(W_T;\mathbb{Z}_T)$	*Generator*
$W(A_3)$		$\mathbb{Z}_3$	α
$W(B_3)$		$\mathbb{Z}_2$	$\alpha = \beta$
$W(H_3)$		0	
$W(I_2(p)) \times W(A_1)$		$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *if p is even* $\mathbb{Z}_2$ *if p is odd*	α, β *if p is even* β *if p is odd*

PROOF. These calculations are in Appendix B. $\square$

PROPOSITION 2.5.25. *When $T = \{s,t\}$,*

$$H_1(W_T;\mathbb{Z}_T) = H_1(I_2(m(s,t));\mathbb{Z}_T) = \mathbb{Z}_{m(s,t)}$$

with generator in the De Concini - Salvetti resolution for W_T given by $\gamma = 1 \otimes \Gamma_s - 1 \otimes \Gamma_t$.

PROOF. This calculation is in Appendix B. $\square$

PROPOSITION 2.5.26. *The first twisted homology of the one generator Coxeter group $W(A_1)$ with generator s is $H_1(W_s;\mathbb{Z}_s) = 0$.*

PROOF. From Example 2.5.6 the twisted resolution has the form

$$\mathbb{Z} \underset{W_s}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_s}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_s}{\otimes} C_0$$

$$\begin{array}{llll} \text{Generators:} & 1 \otimes \Gamma_{s \supset s} & 1 \otimes \Gamma_s & 1 \otimes \Gamma_\emptyset \\ \text{Differentials:} & & 1 \otimes \Gamma_s \longmapsto -2(1 \otimes \Gamma_\emptyset) & \end{array}$$

$$1 \otimes \Gamma_{s \supset s} \longmapsto 0$$

and so the kernel of δ_1 is 0, which completes the proof. $\square$

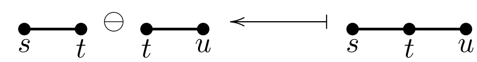
DEFINITION 2.5.27. If the homology of a Coxeter group $H_i(W_T;\mathbb{Z}_T)$ for a group W_T represented by a diagram $\mathcal{D}_{W_T}$ only has one generator, then we represent that generator in the group

$$\bigoplus_{\substack{T \in \mathcal{S} \\ |T| = p}} H_i(W_T;\mathbb{Z}_T)$$

by drawing the diagram $\mathcal{D}_{W_T}$. Suppose W_U is a subgroup of W_T. We represent a non-zero differential in the E^1 page from the generator of $H_i(W_T;\mathbb{Z}_T)$ to the generator of $H_i(W_U;\mathbb{Z}_U)$ by drawing a map from the diagram $\mathcal{D}_{W_T}$ to the diagram $\mathcal{D}_{W_U}$. If the differential is zero, we do not draw the subgroup diagram.

EXAMPLE 2.5.28. We will see in the next proposition that the generator for $H_1(W(A_3);\mathbb{Z}_T)$ is mapped by the transfer map d^1 to the generator for $H_1(W(A_2);\mathbb{Z}_T)$, when $W(A_2)$ is a subgroup of $W(A_3)$. We represent this as:

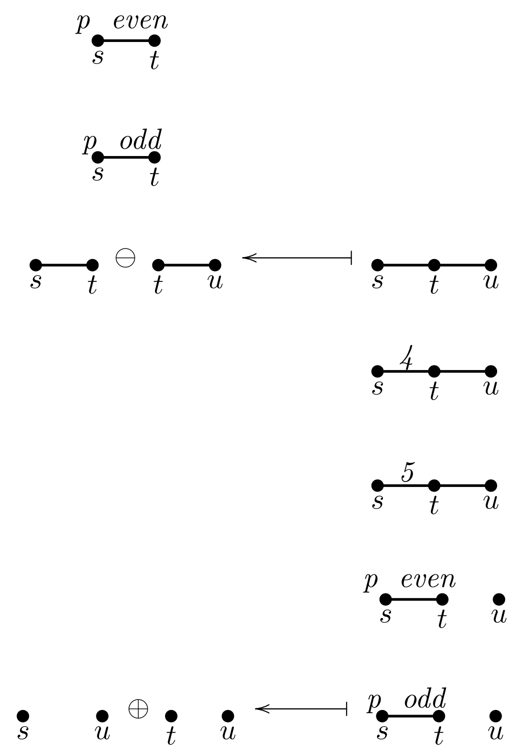
$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T)$$



which shows the diagram A_3 to represent the generator for $H_1(W(A_3);\mathbb{Z}_T)$ when $W(A_3)$ has generating set $\{s,t,u\}$. The two subdiagrams correspond to the generators for $H_1(W(A_2);\mathbb{Z}_T)$ for the two possible $W(A_2)$ subgroups generated by $\{s,t\}$ and $\{t,u\}$. Then this map shows that the generator for H_1 of $W_{\{s,t,u\}}$ maps via the d^1 differential to the generator for H_1 of $W_{\{s,t\}}$ minus the generator for H_1 of $W_{\{t,u\}}$.

PROPOSITION 2.5.29. *The differentials on the E^1 page at $E^1_{2,1}$ are given as in the diagram below, where the diagram notation from Definition 2.5.27 is used. Note here that diagrams representing homology of $W(H_3)$ and $W(I_2(p))\times W(A_1)$ for p even are included, even though their homologies have none and two generators respectively. However the d^1 differential mapping from the homology of either group is zero, and so this does not affect the notation.*

$$\bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T)$$



PROOF. Recall the diagram notation from Definition 2.5.27. This proof involves calculating the differential d^1 (which is the transfer map on each summand by Proposition 2.3.15) on the generators of the homology groups, followed by the collapse map from Definition 2.5.11 which gives the image of this map in terms of the De Concini - Salvetti resolution for the smaller group. These calculations are in Appendix B. $\square$

PROPOSITION 2.5.30. *Recall from Definition 2.1.12 the definition of the diagrams $\mathcal{D}_{\bullet\bullet}$ and $\mathcal{D}_{A_2}$. Then the $E^2_{2,1}$ entry is given by*

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus\Big(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)}\Big).$$

PROOF. Consider the d^1 differentials at $E^2_{2,1}$, as given in Proposition 2.5.29. Applying the splitting technique as in the proof of the $H_2(W;\mathbb{Z})$ calculation (i.e. as in Proposition 2.4.10), we can equate the the kernel of the right hand map over the image of the left hand map to the three summands in the proposition. $\square$

2.5.31. Homology at $E^1_{3,0}$. To calculate this one needs to consider the index of spherical subgroups inside spherical subgroups, as in Section 2.4.4 and in particular Lemma 2.4.6, which gives us that on the bottom row letting the generator of $H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2$ be denoted by 1_T gives the following transfer map, when T' is a subset of T.

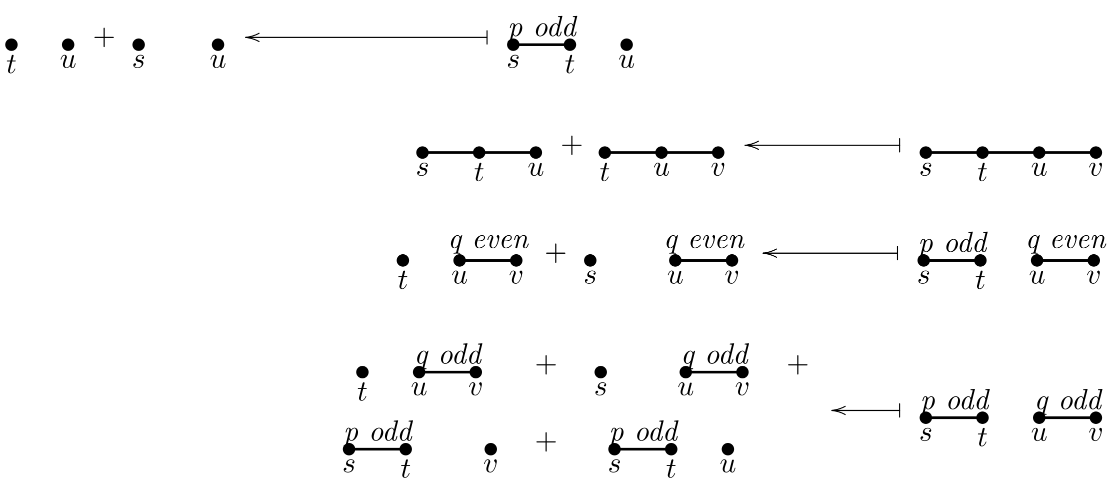
$$\begin{aligned} d^1_{T,T'}: H_0(W_T;\mathbb{Z}_T) &\to H_0(W_{T'};\mathbb{Z}_{T'}) \\ \mathbb{Z}_2 &\to \mathbb{Z}_2 \\ 1_T &\mapsto \begin{cases} 0 & \text{if } |W_T|/|W_{T'}| \text{ is even} \\ 1_{T'} & \text{if } |W_T|/|W_{T'}| \text{ is odd.} \end{cases} \end{aligned}$$

Considering the maps at $E^1_{3,0}$ in the spectral sequence, we have the following

$$\begin{tikzcd} \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=4}} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] \\ \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} \mathbb{Z}_2 & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2 \arrow[l, "d^1"'] & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=4}} \mathbb{Z}_2 \arrow[l, "d^1"'] \end{tikzcd}$$

Lemma 2.5.32. *Recall the notation introduced in Definition 2.5.27, where if the homology of a Coxeter group has one generator, we represent that generator by the corresponding Coxeter diagram. With this notation, the d^1 differentials at $E^1_{3,0}$ are given by the following maps*

$$\begin{tikzcd} \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=4}} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] \\ \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} \mathbb{Z}_2 & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2 \arrow[l, "d^1"'] & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=4}} \mathbb{Z}_2 \arrow[l, "d^1"'] \end{tikzcd}$$



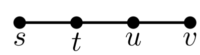
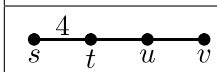
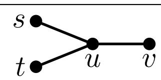
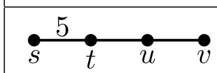
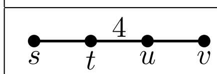
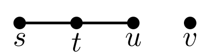
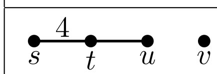
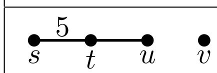
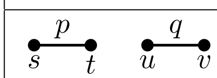
Proof. From Lemma 2.4.9 we know the image of the transfer map

$$d^1 : \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2 \to \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} \mathbb{Z}_2.$$

To compute the transfer map

$$d^1 : \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} \mathbb{Z}_2 \to \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2.$$

we need to consider the index of subgroups with three generators inside finite groups with four generators, by Lemma 2.4.6. This information is displayed in the following table, where p and q are natural numbers greater than or equal to 2:

$\mathcal{D}_W$	$\lvert W_T\rvert$	$\lvert W_{\{s,t,u\}}\rvert$	$\lvert W_{\{s,t,v\}}\rvert$	$\lvert W_{\{s,u,v\}}\rvert$	$\lvert W_{\{t,u,v\}}\rvert$
	120	24	12	12	24
	384	48	16	12	24
	192	24	24	8	24
	14400	120	20	12	24
	1152	48	12	12	48
	48	24	12	8	12
	96	48	16	8	12
	240	120	20	8	12
	$2p\times 2q$	$4p$	$4p$	$4q$	$4q$

Computing the index of each subgroup gives non zero maps as required.

$\square$

Proposition 2.5.33. *Recall from Definition 2.1.12 the definition of the diagrams $\mathcal{D}^{\square}_{\bullet\bullet}$ and $\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}}$ and $\mathcal{D}_{A_3}$. Then the $E^2_{3,0}$ entry on the E^2 page of the spectral sequence is given by*

$$E^2_{3,0} = H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus (\bigoplus_{\substack{W(H_3)\subseteq W\\W(B_3)\subseteq W}} \mathbb{Z}_2)$$

where the sum over $W(H_3)\subseteq W$ and $W(B_3)\subseteq W$ is viewed as a sum over all subsets $I\subset S$ such that W_I is of type B_3 or H_3.

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & A & ? & \cdots & & \\
1 & 0 & 0 & B & ? & \cdots & \\
0 & \mathbb{Z} & ? & ? & C & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 &
\end{array}
$$

FIGURE 2. The E^2 page of the isotropy spectral sequence for the Davis complex of a Coxeter group W.

PROOF. Splitting the d^1 differentials of Lemma 2.5.32 as in the proof of the $H_2(W;\mathbb{Z})$ calculation (i.e. as in Proposition 2.4.10), we can equate the kernel of the left hand differentials over the image of the right hand differentials to the components on the right hand side of the above expression. This gives the formula for the E^2 term as required. $\square$

2.5.34. Further differentials are zero. Recall the isotropy spectral sequence for the Davis complex associated to a group W, given in Figure 1. Then from the calculations of $E^2_{i,j}$ for the diagonal $i+j=2$ in Section 2.4 and the diagonal $i+j=3$ in the previous four subsections, the spectral sequence has E^2 page as shown in Figure 2.

Where A is $H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$,

B is $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)})$

and C is $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)\oplus(\bigoplus_{\substack{W(H_3)\subseteq W\\ W(B_3)\subseteq W}}\mathbb{Z}_2)$.

The E^{∞} page of this spectral sequence gives us filtration quotients for $H_3(W;\mathbb{Z})$ (up to extension) on the blue diagonal. The argument in this section shows that all possible further differentials to and from the blue diagonal are zero. Since the spectral sequence is first quadrant, all possible further differentials out from the groups A and B are zero, and one can see from the diagram that the possible d^2 and d^3 differentials from C also have target groups 0. Therefore there are only 3 possible further differentials that may affect the blue diagonal:

(1) $d^2: E^2_{3,1}\to A$
(2) $d^2: E^2_{4,0}\to B$
(3) $d^3: E^3_{4,0}\to E^3_{1,2}$.

To compute these differentials we first prove two lemmas which will reduce the cases for which we compute $E^2_{4,0}$.

Denote the isotropy spectral sequence $E(A\times B)$ for Coxeter group $W_A\times W_B$, where W_A and W_B are non trivial finite groups, and the size of their generating sets add to 4. Then the $E^1_{4,0}$ term in the spectral sequence is

$$E^1_{4,0}=H_0(W_A\times W_B;\mathbb{Z}_{A\sqcup B}).$$

Lemma 2.5.35. *With notation as above, the possible d^2 and d^3 differentials originating at $E^r_{4,0}$, for $r=2$ or $r=3$, in the spectral sequence $E(A\times B)$ are zero.*

Proof. By the Künneth theorem for group homology (see e.g. [**12**]) we have the short exact sequence:

$$0\to\bigoplus_{i+j=k}H_i(W_A;\mathbb{Z}_A)\otimes_{\mathbb{Z}}H_j(W_B;\mathbb{Z}_B)\xrightarrow{\times}H_k(W_A\times W_B;\mathbb{Z}_{A\sqcup B})$$
$$\to\bigoplus_{i+j=k-1}Tor_1^{\mathbb{Z}}(H_i(W_A;\mathbb{Z}_A),H_j(W_B;\mathbb{Z}_B))\to$$

since $\mathbb{Z}_A\otimes\mathbb{Z}_B\cong\mathbb{Z}_{A\sqcup B}$, and when $k=0$ we have

$$\begin{aligned}&\bigoplus_{i+j=k-1}Tor_1^{\mathbb{Z}}(H_i(W_A;\mathbb{Z}_A),H_j(W_B;\mathbb{Z}_B))\\ =\ &\bigoplus_{i+j=-1}Tor_1^{\mathbb{Z}}(H_i(W_A;\mathbb{Z}_A),H_j(W_B;\mathbb{Z}_B))\\ =\ &0\end{aligned}$$

hence the short exact sequence gives

$$H_0(W_A;\mathbb{Z}_A)\otimes_{\mathbb{Z}} H_0(W_B;\mathbb{Z}_B) \xrightarrow{\cong} H_0(W_A\times W_B;\mathbb{Z}_{A\sqcup B}).$$

By Theorem 2.3.19 there is a pairing

$$\Phi_* : E(A)\otimes E(B)\to E(A\times B)$$

which is given on individual summands by the Künneth map. Therefore for $E^1_{4,0}$ (since it has only one summand) Φ_* is given by the Künneth map above, which is an isomorphism. Under the pairing Φ_* all cycles in $E^1_{4,0}$ in $E(A\times B)$ therefore correspond to a pair of cycles: one in $E^1_{p,0}$ in $E(A)$ and one in $E^1_{4-p,0}$ in $E(B)$. Since moving from page E^r to page E^{r+1} calculates homology with respect to d^r, cycles in $E^2_{4,0}$ in $E(A\times B)$ will be quotients of cycles in $E^1_{4,0}$ in $E(A\times B)$, and cycles in $E^3_{4,0}$ in $E(A\times B)$ will be quotients of these.

Under Φ_* the differentials satisfy a Leibniz rule: in the image of the pairing the differentials d^r for the spectral sequence $E(A\times B)$ can be written in terms of the differentials d^r for the spectral sequence $E(A)$ and the spectral sequence $E(B)$. Since all cycles in $E^r_{4,0}$ for $r=1,2,3$ in $E(A\times B)$ are defined via Φ_* on the E^1 page, it follows that the differentials d^r originating at these positions are defined purely in terms of the differentials d^r in $E(A)$ and $E(B)$ originating at this position, via a Leibniz rule.

Since the number of generators in W_A or in W_B is less than the number of generators in $W_A\times W_B$, the differentials in $E(A)$ and $E(B)$ that occur in this Leibniz rule will originate at $E^r_{p,0}$ where $p<4$. But all possible targets of a d^2 or d^3 differential from such an $E^r_{p,0}$ are zero, since they are zero on the E^2 page of both $E(A)$ and $E(B)$ (consider the spectral sequence in Figure 2). Thus the further differentials mapping from $E^r_{4,0}$ in $E(A\times B)$ are zero. $\square$

LEMMA 2.5.36. *Consider a differential d^2 or d^3 originating from a summand in $E^r_{4,0}$ for $r=2$ or $r=3$, in the isotropy spectral sequence for a Coxeter group W. If the corresponding cycle at the $E^1_{4,0}$ term is a summand $H_0(W_A\times W_B;\mathbb{Z}_{A\sqcup B})$, for W_A and W_B non-trivial subgroups of W, then the d^2 or d^3 differential is zero.*

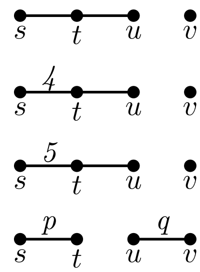
PROOF. By Lemma 2.3.16, the inclusion of groups $W_A\times W_B\hookrightarrow W$ gives an inclusion of spectral sequences on the E^1 page

$$E^1(A\times B)\hookrightarrow E^1(W).$$

Therefore differentials mapping from cycles corresponding to the $H_0(W_A\times W_B;\mathbb{Z}_{A\sqcup B})$ summand at position $E^1_{4,0}$ in $E(W)$ will be given by differentials in $E(A\times B)$.

From Lemma 2.5.35 the d^2 and d^3 differentials originating at the $E^r_{4,0}$ position are zero in $E(A\times B)$. This completes the proof. $\square$

COROLLARY 2.5.37. *Consider d^2 and d^3 differentials originating at summands in $E^2_{4,0}$ and $E^3_{4,0}$. If the corresponding cycles at the $E^1_{4,0}$ term come from $H_0(W_T;\mathbb{Z}_T)$ such that W_T is one* *of the following groups, then the d^2 and d^3 differentials are zero. Below p and q are integers greater than or equal to 2.*



We therefore only need to consider the $E^2_{4,0}$ components which come from the

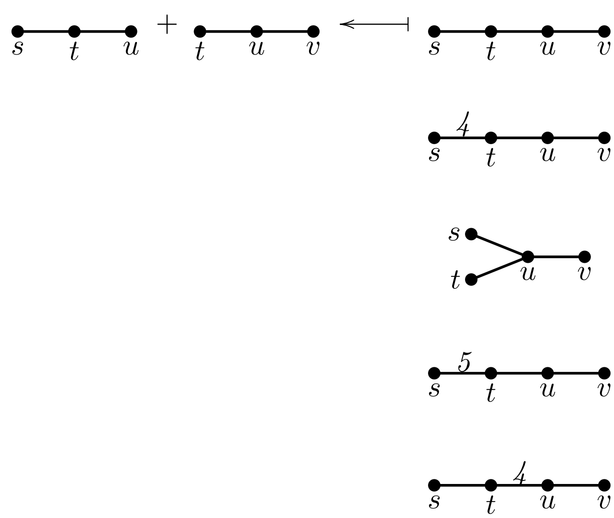
$$E^1_{4,0} = \bigoplus_{\substack{T \in \mathcal{S} \\ |T|=4}} H_0(W_T;\mathbb{Z}_T)$$

cycles relating to the groups which do not arise as products, namely for W_T of type A_4, B_4, D_4, F_4 and H_4. Recall that all Coxeter groups satisfy $H_0(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ and recall the notation of Definition 2.5.27 which allows us to represent this homology class by the corresponding Coxeter diagram.

Lemma 2.5.38. *With notation as above, the differentials on the E^1 page at the $E^1_{4,0}$ position for the summands $H_0(W_T;\mathbb{Z}_T)$ corresponding to Coxeter groups of type A_4, B_4, D_4, F_4 and H_4 have the following form:*

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=5}} H_0(W_T;\mathbb{Z}_T)$$

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2 \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} \mathbb{Z}_2 \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=5}} \mathbb{Z}_2$$



Proof. From Lemma 2.5.32 we have the maps from the central groups to the left. The finite Coxeter groups with 5 generators for which the A_4, B_4, D_4, F_4 and H_4 diagrams are subdiagrams are the groups of type A_5, B_5, D_5 and the product groups of type $A_4 \times A_1, B_4 \times A_1, D_4 \times A_1, F_4 \times A_1$ and $H_4 \times A_1$. Recall from Lemma 2.4.6 that the transfer map on the bottom row is determined by the index of the subgroup. In the case of the product groups, the index of the corresponding 4-generator subgroup is 2 and hence the transfer map is zero. We are therefore left with the following computations:

- $|W(A_4)| = 120$, $|W(A_5)| = 720$ so $|W(A_5) : W(A_4)| = 6$
- $|W(B_4)| = 384$, $|W(B_5)| = 3840$ so $|W(B_5) : W(B_4)| = 10$
- $|W(D_4)| = 192$, $|W(D_5)| = 1920$ so $|W(D_5) : W(D_4)| = 10$

which we compute using Python and [**26**], though formulas for each group size can be found in [**33**]. Since in each case the index of the subgroup is even, the transfer map is zero. $\square$

Proposition 2.5.39. *If the d^1 differential originating at a summand $H_q(W_T;\mathbb{Z}_T)$ on the E^1 page of the isotropy spectral sequence is identically zero on the chain level, then the higher differentials which originate at cycles corresponding to $H_q(W_T;\mathbb{Z}_T)$ on the E^r page are also zero.*

PROOF. The d^1 differential of the isotropy spectral sequence is given by the transfer map on the chain level by Proposition 2.3.15. In general higher differentials of the spectral sequence for a double complex are induced by combinations of the differentials on the chain level, and lifting on the chain level. For example given a double complex $C_{p,q}$ the d^2 differential is induced on the chain level as follows:

$$\begin{tikzcd}
C_{p-2,q+1} & C_{p-1,q+1} \arrow[l] \arrow[d] & \\
& C_{p-1,q} \arrow[u, dotted, bend left, "lift"] & C_{p,q} \arrow[l] \arrow[ull, bend right, "d^2"']
\end{tikzcd}$$

Therefore if the d^1 differential is zero on the chain level for the cycle representing a term $E^r_{p,q}$, then the higher differentials will also be zero. □

COROLLARY 2.5.40. *The d^2 and d^3 differentials originating at the $E^r_{0,4}$ position for $r = 2$ or $r = 3$ corresponding to cycles on the $E^1_{4,0}$ summands for groups of type B_4, D_4, F_4 and H_4 are zero.*

PROOF. This is a consequence of Lemma 2.5.38, and Proposition 2.5.39, if we prove that the transfer maps given in Lemma 2.4.6 on the chain level originating at $H_0(W_T;\mathbb{Z}_T)$ for these groups are identically zero (and not just zero modulo 2). This is satisfied if, alongside there being an even number of cosets, there are identical numbers of cosets with odd and even length. Then the transfer map on the chain level for C_0:

$$\begin{aligned} tr : \mathbb{Z}_{T'} &\to \mathbb{Z}_T \\ m &\mapsto \sum_{g \in W_{T'} \backslash W_T} g \cdot m \end{aligned}$$

will map identically to zero, since the coset acts on m as the identity if it has even length and negation if it has odd length. Using Python [**26**] we write a short program which returns the number of coset representatives of even and odd length, given a group and a subgroup. The code can be found in Appendix A. We then compute that in the cases of B_4, D_4, F_4 and H_4, every three generator subgroup has an equal number of even length and odd length cosets. Therefore they transfer identically to zero, so we can apply Proposition 2.5.39. □

We are therefore left with a potential d^2 or d^3 differential originating at the $E^r_{0,4}$ position for $r = 2$ or $r = 3$, corresponding to cycles on the $E^1_{4,0}$ summand $H_0(W(A_4);\mathbb{Z}_T)$. This summand is non-zero when $W(A_4)$ arises as a spherical subgroup of W. We compute the spectral sequence for $W(A_4)$ and note by Lemma 2.3.16 that any further differentials occurring in the spectral sequence for W corresponding to this summand, will occur in the spectral sequence for $W(A_4)$, via the inclusion of $W(A_4)$ into W.

LEMMA 2.5.41. *The potential d^2 and d^3 differentials originating at the $E^r_{0,4}$ position for $r = 2$ or $r = 3$ and corresponding to cycles on the $E^1_{4,0}$ summand $H_0(W(A_4);\mathbb{Z}_T)$ are zero.*

PROOF. If the further differentials were non zero then they would also be non zero in the spectral sequence for $W(A_4)$ by Lemma 2.3.16. The E^2 page for the Coxeter group $W(A_4)$ is given by

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_3 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 &
\end{array}
$$

and the computation of this is given in Appendix B. Therefore the blue diagonal in the spectral sequence contains the groups $\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_2$. The third integral homology of the symmetric group on 5 letters, which corresponds to $W(A_4)$, is

$$H_3(W(A_4);\mathbb{Z}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$$

which is precisely given by letting the groups on the blue diagonal be the E^∞ terms, or *filtration quotients* for $H_3(W(A_4);\mathbb{Z})$ (here there is a non-trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ to get $\mathbb{Z}_4$ which we will discuss in the following section). Therefore the E^2 page is equal to the E^∞ page on the blue diagonal, and so no higher differentials in or out of this diagonal are are non-zero. $\square$

PROPOSITION 2.5.42. *The possible d^2 and d^3 originating at the $E^*_{4,0}$ group in the spectral sequence are zero.*

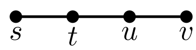
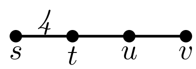
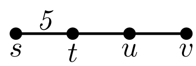
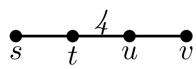
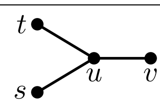
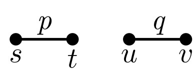
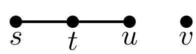
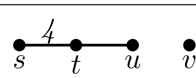
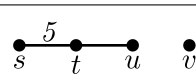
PROOF. This is direct result of putting together Corollaries 2.5.37 and 2.5.40 and Lemma 2.5.41. $\square$

To compute the potential d^2 differential from $E^2_{3,1}$ to $E^2_{1,2}$, we first compute the $E^2_{3,1}$ term in the spectral sequence.

LEMMA 2.5.43. *We have the following first homology groups $H_1(W_T;\mathbb{Z}_T)$ for finite Coxeter groups with 4 generators. Generators are given by the De Concini - Salvetti resolution for*

W_T: we let

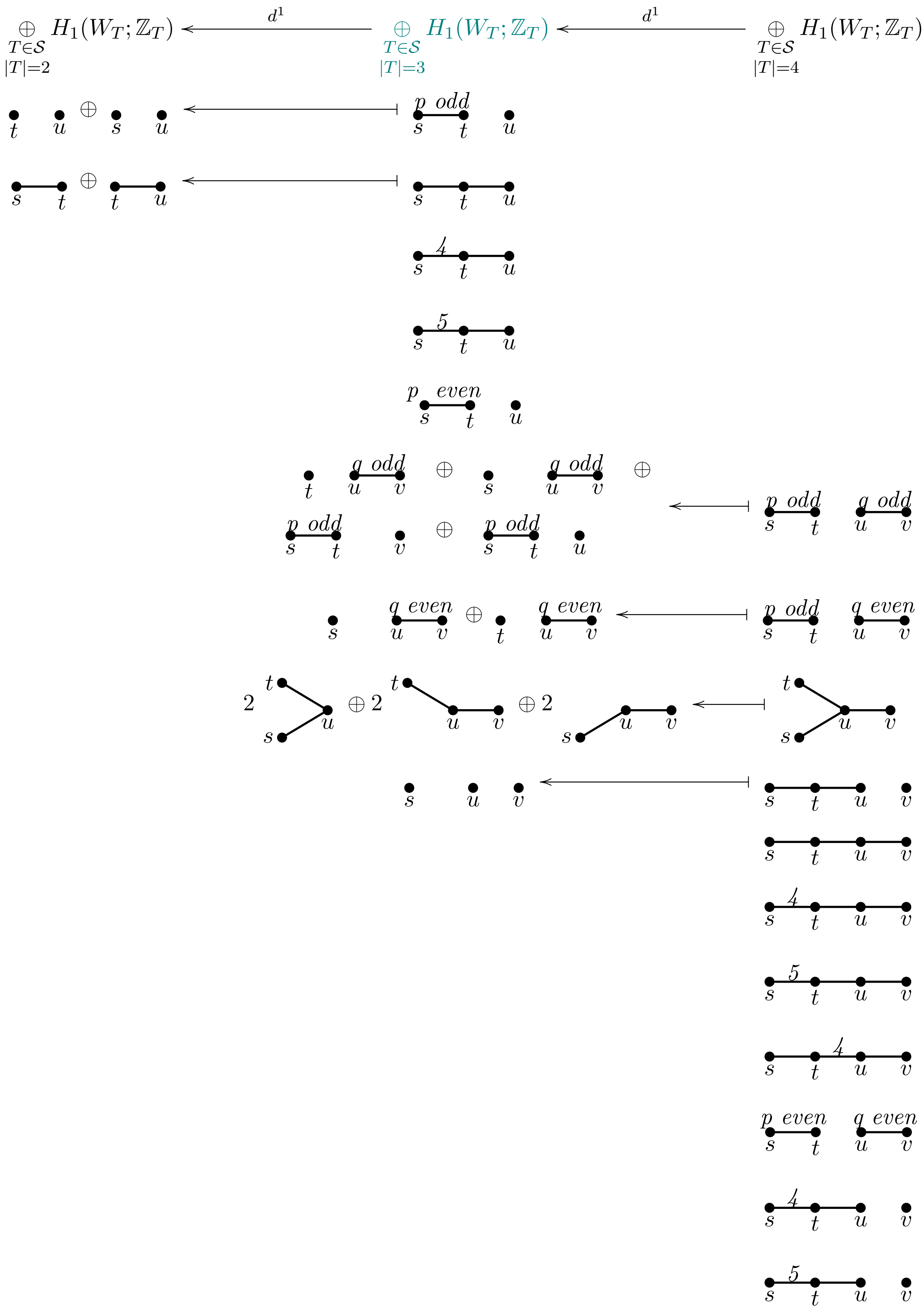
$$\begin{aligned}\alpha &= (1\otimes\Gamma_s)-(1\otimes\Gamma_t)\ ,\\ \beta &= (1\otimes\Gamma_s)-(1\otimes\Gamma_u),\\ \gamma &= (1\otimes\Gamma_s)-(1\otimes\Gamma_v).\end{aligned}$$

W_T	$\mathcal{D}_{W_T}$	$H_1(W_T;\mathbb{Z}_T)$	*Generators*
$W(A_4)$		0	
$W(B_4)$		$\mathbb{Z}_2$	$\alpha=\beta=\gamma$
$W(H_4)$		0	
$W(F_4)$		$\mathbb{Z}_2$	$\beta=\gamma$
$W(D_4)$		$\mathbb{Z}_3$	β
$W(I_2(p))\times W(I_2(q))$		$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ p,q *even* $\mathbb{Z}_2\oplus\mathbb{Z}_2$ p *odd*, q *even* $\mathbb{Z}_2\oplus\mathbb{Z}_2$ p *even*, q *odd* $\mathbb{Z}_2$ p,q *odd*	α,β,γ β,γ $\alpha,\beta=\gamma$ $\beta=\gamma$
$W(A_3)\times W(A_1)$		$\mathbb{Z}_2$	γ
$W(B_3)\times W(A_1)$		$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\alpha=\beta,\ \gamma$
$W(H_3)\times W(A_1)$		$\mathbb{Z}_2$	γ

Proof. These calculations are in Appendix B. □

Lemma 2.5.44. *Recall the notation introduced in Definition 2.5.27, where if the homology of a Coxeter group has one generator, we represent that generator by the corresponding Coxeter diagram. Using this notation, the d^1 differentials on the E^1 page at the position $E^1_{1,3}$ are given by the following maps. Here we note that some of the groups satisfy that the homology has two or more generators. In all but one case these generators all map to zero, which is shown by no map originating at the diagram. In the isolated case $I_2(p)\times I_2(q)$ where p is odd and q*

is even, the two generators are mapped to the two generators for the subgroups shown by the identity map.


$\bigoplus_{T\in\mathcal{S},\,|T|=2} H_1(W_T;\mathbb{Z}_T)$
d^1
$\bigoplus_{T\in\mathcal{S},\,|T|=3} H_1(W_T;\mathbb{Z}_T)$
d^1
$\bigoplus_{T\in\mathcal{S},\,|T|=4} H_1(W_T;\mathbb{Z}_T)$

PROOF. The left hand maps are given by Proposition 2.5.29. We compute the transfer and collapse maps on the right using Python, as in the sample Example A.1 in Appendix A. These calculations are in Appendix B. □

Denote the isotropy spectral sequence $E(T\times V)$ for Coxeter group $W_T\times W_V$, where W_T and W_V are non trivial finite groups, and the size of their generating sets add to 3.

LEMMA 2.5.45. *With notation as above, the possible d^2 differential originating at $E^2_{3,1}$, in the spectral sequence $E(T\times V)$ is zero.*

PROOF. Note that by the Künneth theorem for groups:

$$\begin{aligned} H_1(W_T\times W_V;\mathbb{Z}_{T\cup V}) \;&\cong\; (H_1(W_T;\mathbb{Z}_T)\otimes H_0(W_V;\mathbb{Z}_V))\oplus(H_0(W_T;\mathbb{Z}_T)\otimes H_1(W_V;\mathbb{Z}_V)) \\ &\quad \oplus Tor_1^{\mathbb{Z}}(H_0(W_T;\mathbb{Z}_T),H_0(W_V;\mathbb{Z}_V)) \end{aligned}$$

By Theorem 2.3.19 if the d^2 originates from either the $(H_1(W_T;\mathbb{Z}_T)\otimes H_0(W_V;\mathbb{Z}_V))$ component or the $(H_0(W_T;\mathbb{Z}_T)\otimes H_1(W_V;\mathbb{Z}_V))$ component of the right hand side of the isomorphism, it is in the image of the pairing

$$\Phi_*: E(T)\otimes E(V)\to E(T\times V)$$

which is given by the Künneth map on components.

In the image of Φ_*, the d^2 differential on the left hand side satisfies a Leibniz rule. That is the d^2 differential on $E(T\times V)$ is determined by the d^2 differentials on $E(T)$ and the d^2 differentials on $E(V)$. By similar reasoning as in the proof of Lemma 2.5.35 these differentials are zero, and therefore via the Leibniz rule the d^2 originating at a cycle in the image of Φ_* is zero.

It remains to show that a d^2 differential originating at a cycle corresponding to the Tor summand of the right hand side of the Künneth isomorphism at $E^1_{3,1}$ in $E(T\times V)$ is zero. That is, the group $Tor_1^{\mathbb{Z}}(H_0(W_T;\mathbb{Z}_T),H_0(W_V;\mathbb{Z}_V))=\mathbb{Z}_2$ and there may exist a d^2 differential corresponding to a map originating at this $\mathbb{Z}_2$. Consider the following short exact sequence:

$$H_1(W_T\times W_V;\mathbb{Z}_{T\cup V})\overset{\times 2}{\to} H_1(W_T\times W_V;\mathbb{Z}_{T\cup V})\overset{\rho_2}{\to} H_1(W_T\times W_V;\mathbb{Z}_2)$$

where ρ_2 is mod 2 reduction. The class corresponding to Tor (let's call it α) in the middle summand will satisfy $\rho_2(\alpha)\neq 0$, since it represents 2-torsion, but by the Künneth formula,

$$H_1(W_T\times W_V;\mathbb{Z}_2)\cong(H_1(W_T;\mathbb{Z}_2)\otimes H_0(W_V;\mathbb{Z}_2))\oplus(H_0(W_T;\mathbb{Z}_2)\otimes H_1(W_V;\mathbb{Z}_2)).$$

Therefore, if we consider the isotropy spectral sequence for $W_T\times W_V$, but with $\mathbb{Z}_2$ coefficients, i.e. the sequence for $H_*(W_T\times W_V;\mathbb{Z}_2)$, by the pairing of spectral sequences in Theorem 2.3.19 and the same reasoning as the proof of Lemma 2.5.35, the class corresponding to $\rho_2(\alpha)$ will be mapped to zero under the d^2 differential: $d^2(\rho_2(\alpha))=0$. However the target of the differential is all 2-torsion (it is given by $H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$) and so this survives in the reduction ρ_2. Since the

d^2 differential commutes with mod 2 reduction, computing d^2 on α and then reducing should give the zero map, i.e.

$$\rho_2(d^2(\alpha)) = d^2(\rho_2(\alpha)) = 0.$$

Since the target is unchanged by reduction, $\rho_2(d^2(\alpha)) = d^2(\alpha)$ and so $d^2(\alpha)$ must be zero. $\square$

LEMMA 2.5.46. *Suppose a d^2 differential in the isotropy spectral sequence for W originates at a cycle in $E^2_{3,1}$ represented by a homology class in $E^1_{3,1}$ of a subgroup $W_T \times W_V$ of W such that neither W_T or W_V is the trivial group. Then this d^2 differential is the zero map.*

PROOF. By Lemma 2.3.16, the inclusion of groups $W_T \times W_V \hookrightarrow W$ gives an inclusion of spectral sequences on the E^1 page

$$E^1(T \times V) \hookrightarrow E^1(W)$$

such that in the image of the inclusion the differentials in $E(T \times V)$ give the differentials in $E(W)$. Therefore all cycles corresponding to the $H_1(W_T \times W_V; \mathbb{Z}_{T \sqcup V})$ summand at position $E^1_{3,1}$ in $E(W)$ will be given by differentials in $E(A \times B)$.

From Lemma 2.5.45 the possible d^2 differential originating at $E^2_{3,1}$, in the spectral sequence $E(T \times V)$ is zero. This completes the proof. $\square$

PROPOSITION 2.5.47. *The possible d^2 differential originating at the $E^2_{3,1}$ group in the spectral sequence is zero.*

PROOF. The $E^2_{3,1}$ entry is calculated by computing the homology of the sequence given in Lemma 2.5.44. Its origin is therefore cycles in summands of the form $H_1(W_T; \mathbb{Z}_T)$ for $|T| = 3$. Note that the target of this d^2 differential is given by $E^2_{1,2} = H_0(\mathcal{D}_{odd}; \mathbb{Z}_2)$, which is all two torsion.

If the origin of the d^2 map is a cycle in the summand $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_3$ for $W_T = W(A_3)$, it must map via d^2 to zero, since the target is all 2-torsion and the source is 3-torsion.

If the origin of the d^2 map is a cycle in the summand $H_1(W_T; \mathbb{Z}_T)$ for $W_T = W(B_3)$, $W_T = W(H_3)$ or $W_T = W(I_2(p)) \times W(A_1)$ it will map to zero, as the representing cycles transfer identically to zero on the chain level by the proof of Lemma 2.5.44, so we can apply Proposition 2.5.39.

Lemma 2.5.46 covers the final cases where the d^2 originates at a cycle in the summand $H_1(W_T; \mathbb{Z}_T)$ for $W_T = W(I_2(p)) \times W(A_1)$ for $2 \leq p$. $\square$

2.5.48. Extension problems. Recall the isotropy spectral sequence for the Davis complex associated to a group W, given in Figure 1. Then from the calculations of $E^2_{i,j}$ for the diagonal $i + j = 2$ in Section 2.4, the diagonal $i + j = 3$ in this section, and since all further differentials with target or source group on the blue diagonal are zero from the previous subsection, the spectral sequence has the following E^∞ page.

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & A & ? & \cdots & & \\
1 & 0 & 0 & B & ? & \cdots & \\
0 & \mathbb{Z} & ? & ? & C & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 &
\end{array}
$$

Where A is $H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$,

B is $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)})$

and C is $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{\bullet\ \bullet\overset{\text{even}}{-}\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)\oplus(\bigoplus_{\substack{W(H_3)\subseteq W\\ W(B_3)\subseteq W}}\mathbb{Z}_2)$.

And the spectral sequence on this diagonal converges to $H_3(W;\mathbb{Z})$, so we are left to consider possible extensions on this diagonal. That is there is a filtration of $H_3(W;\mathbb{Z})$

$$F_0\subseteq F_1\subseteq F_2\subseteq F_3=H_3(W;\mathbb{Z})$$

where $E^\infty_{0,3}=F_0$, $E^\infty_{1,2}=F_1/F_0$, $E^\infty_{2,1}=F_2/F_1$ and $E^\infty_{3,0}=F_3/F_2$. In our case we have $F_0=0$ and so $E^\infty_{1,2}=H_0(\mathcal{D}_{odd};\mathbb{Z}_2)=F_1$.

Proposition 2.5.49. *We have that $F_1=A=H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$ splits off by an analogue of the sign homomorphism for symmetric groups.*

Proof. Consider a homomorphism ψ from a Coxeter group W with generating set S to the cyclic subgroup of order two generated by t in S, which is isomorphic to $W(A_1)$. If two generators of W, s_1 and s_2, satisfy $m(s_1,s_2)$ is odd then we require $\psi(s_1)=\psi(s_2)$, whereas if $m(s_1,s_2)$ is even there is no requirement on ψ. A summand of

$$A=F_1=H_0(\mathcal{D}_{odd};\mathbb{Z}_2)=\bigoplus_{\pi_0(\mathcal{D}_{odd})}\mathbb{Z}_2$$

is represented by a vertex of $\mathcal{D}(W)$. For the vertex t generating the subgroup $W(A_1)$, denote the corresponding summand of A by $\mathbb{Z}_2(t)$. We define the homomorphism ψ from W to $W(A_1)$ to be zero on all but one of the connected components of $\mathcal{D}_{odd}$, namely the t component.

$$\begin{aligned}\psi : W &\to W(A_1)\\ s &\mapsto \begin{cases} t & \text{if } s \text{ and } t \text{ are in the same component of } \pi_0(\mathcal{D}_{odd})\\ e & \text{otherwise.}\end{cases}\end{aligned}$$

Then the map ψ induces a map ψ_* which fits into the following diagram

$$\begin{tikzcd}
\mathbb{Z}_2(t) \arrow[r, hook] \arrow[drrr, "id"'] & A \arrow[r, hook] & H_3(W;\mathbb{Z}) \arrow[r, "\psi_*"] & H_3(W(A_1);\mathbb{Z}) \arrow[d, equal] \\
 & & & \mathbb{Z}_2
\end{tikzcd}$$

where $H_3(W(A_1);\mathbb{Z}) = \mathbb{Z}_2$ is computed by noting that the E^∞ page of the isotropy spectral sequence for $W(A_1)$ has only one group on the blue diagonal: the $H_0(\mathcal{D}_{odd};\mathbb{Z}_2)$ component corresponding to t ($\mathbb{Z}_2(t)$). The inclusion map $A \hookrightarrow H_3(W;\mathbb{Z})$ comes from the fact that A is at the top left of the diagonal of filtration quotients for W, and so is a subgroup of $H_3(W;\mathbb{Z})$. The isomorphism gives us that $H_3(W;\mathbb{Z})$ splits as

$$H_3(W;\mathbb{Z}) = \mathbb{Z}_2(t) \oplus ker(\psi_*)$$

and so there are no extensions involving the $\mathbb{Z}_2(t)$ summand of A. Repeating this argument over all summands gives that there are no extensions involving A and so $A = F_1$ splits off in $H_3(W;\mathbb{Z})$, as required.

$\square$

We therefore have the filtration

$$0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = H_3(W;\mathbb{Z}) = F_1 \oplus F_3'$$

and so $F_2 = F_1 \oplus F_2'$ and $F_3 = F_1 \oplus F_3'$. It follows that $E^\infty_{2,1} = B = F_2/F_1 = F_2'$ and $E^\infty_{0,3} = C = F_3/F_2 = F_3'/F_2'$, so F_3' fits into the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & F_2' \arrow[r] \arrow[d, equal] & F_3' \arrow[r] \arrow[d, equal] & F_3'/F_2' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & F_3' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

i.e. F_3' is an extension of C by B.

LEMMA 2.5.50. *There exist no non-trivial extensions between $H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2)$ in C and B.*

PROOF. A summand of $H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2)$ is represented by a vertex in $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ which is given by an $I_2(2p) \sqcup A_1$ subdiagram present in $\mathcal{D}_W$. We compute the spectral sequence for the Coxeter group $V = W(I_2(2p)) \times W(A_1)$ corresponding to this diagram, and note that by Lemma 2.3.16 the inclusion of the subgroup V into the group W induces a map of spectral sequences. Therefore if there is a trivial extension in the spectral sequence for V corresponding to the $I_2(2p) \sqcup A_1$ summand of $H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!\!-\!\!\bullet}};\mathbb{Z}_2)$, this extension will be trivial in the spectral sequence for W. This is because the splitting of the extension sequence in $E(V)$ will give a splitting of the extension sequence in $E(W)$, under the map of spectral sequences.

We consider first the case when $p > 1$ and then the case $p = 1$. The E^∞ page for the Coxeter group $V = W(I_2(2p)) \times W(A_1)$, for $p > 1$ is given by

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

which is computed in Appendix B. We therefore have that $H_3(V;\mathbb{Z}) = F_3' \oplus F_1 = F_3' \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ where F_3' is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$.

The third integral homology of $V = W(I_2(2p)) \times W(A_1)$ can be computed via the Künneth formula for groups, to be

$$H_3(W(I_2(2p)) \times W(A_1);\mathbb{Z}) \quad = \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

We compute this in Appendix B.
Therefore we see that $F_3' = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$ and it follows that there is no non-trivial extension between the $H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!\!-\!\!\bullet}};\mathbb{Z}_2)$ component of C and B. For the case $p = 1$, i.e. $V =$

$W(I_2(p)) \times W(A_1) = W(A_1) \times W(A_1) \times W(A_1)$, we have the following E^∞ page:

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

which is computed in Appendix B.
We therefore have that $H_3(V;\mathbb{Z}) = F_3' \oplus F_1 = F_3' \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ where F_3' is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
The third integral homology of $V = W(A_1) \times W(A_1) \times W(A_1)$ is given by that of $W(I_2(2p)) \times W(A_1)$ when $p = 1$ and from the previous calculation is therefore:

$$H_3(W(A_1) \times W(A_1) \times W(A_1);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Therefore we see that $F_3' = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and it follows that there is no non-trivial extension between the $H_0(\mathcal{D}_{\bullet\ \bullet\overset{\text{even}}{-}\bullet};\mathbb{Z}_2)$ component of C and B. $\square$

Lemma 2.5.51. *There exists a non-trivial extension between the $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ component in C and the $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ component in B.*

Proof. A summand of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ is represented by a vertex of $\mathcal{D}_{A_3}$, which is given by an A_3 subdiagram present in $\mathcal{D}_W$. We compute the spectral sequence for the subgroup $V = W(A_3)$ corresponding to this diagram. The E^∞ page for the Coxeter group $V = W(A_3)$ is given by

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_3 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

which is computed in Appendix B. We therefore have $H_3(V;\mathbb{Z}) = F_3' \oplus F_1 = F_3' \oplus \mathbb{Z}_2$ where F_3' is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. Recall that $V = W(A_3)$ is the symmetric group S_4. The third integral homology of $V = W(A_3)$ is $H_3(S_4;\mathbb{Z}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$ and the unique extension which will obtain this result is the following:

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3 \to \mathbb{Z}_4 \oplus \mathbb{Z}_3 \to \mathbb{Z}_2 \to 0$$

giving $H_3(V;\mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$. By Lemma 2.3.16 the inclusion of subgroup V into group W gives a map of spectral sequences. Under this map the extension sequence above is mapped as follows.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \arrow[r] \arrow[d, hook] & \mathbb{Z}_4 \oplus \mathbb{Z}_3 \arrow[r] \arrow[d, dotted] & \mathbb{Z}_2 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & B \arrow[r] & F_3' \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}
$$

Therefore the extension in the V spectral sequence corresponding to the A_3 summand of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ is present in the spectral sequence for W. It follows that there exists a non trivial extension from each summand of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ to $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$. $\square$

Definition 2.5.52. For a Coxeter group W, let $I = \pi_0(\mathcal{D}_{\bullet\bullet})$, $J = \pi_0(\mathcal{D}_{A_3})$ and let the connected component of a vertex $\{s,u\}$ in $\pi_0(\mathcal{D}_{\bullet\bullet})$ be denoted $[\{s,u\}]$ and the connected component of a vertex $\{s,t,u\}$ in $\pi_0(\mathcal{D}_{A_3})$ be denoted $[\{s,t,u\}]$. We define the *extension matrix* X_W to be the I by J matrix with entries

$$X(i,j) = \begin{cases} 1 & \text{if } i = [\{s,u\}] \text{ and } j = [\{s,t,u\}] \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.5.53. *Given a Coxeter group W, the extension of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ in the spectral sequence is completely determined by the extension matrix X_W defined in Definition 2.5.52.*

PROOF. For two finite indexing sets I and J, the extensions of $\bigoplus_J \mathbb{Z}_2$ by $\bigoplus_I \mathbb{Z}_2$ are classified by

$$\begin{aligned} \operatorname{Ext}(\bigoplus_I \mathbb{Z}_2, \bigoplus_J \mathbb{Z}_2) &= \bigoplus_I \bigoplus_J \operatorname{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) \\ &= \bigoplus_I \bigoplus_J \mathbb{Z}_2. \end{aligned}$$

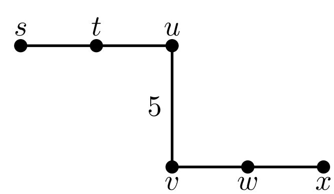
Under this classification, an extension is given by a tuple of entries, either zero or 1, for each pair (i,j) in $I\times J$. The (i,j) entry is zero if the restriction to these summands in the extension sequence is a trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ ($\mathbb{Z}_2\oplus\mathbb{Z}_2$), and 1 if the extension is the non-trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ ($\mathbb{Z}_4$). Letting the (i,j) entry in the tuple be $X(i,j)$ gives an $I\times J$ matrix X.

The extension of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ is given by the following extension sequence

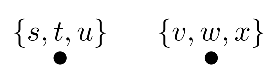
$$\begin{tikzcd} 0 \arrow[r] & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bigoplus_{\pi_0(\mathcal{D}_{\bullet\bullet})} \mathbb{Z}_2 \arrow[r] & Y \arrow[r] & \bigoplus_{\pi_0(\mathcal{D}_{A_3})} \mathbb{Z}_2 \arrow[r] & 0. \end{tikzcd}$$

By Lemma 2.5.51, we know that the restriction on the right to a $\mathbb{Z}_2$ summand with index a class of vertices $[\{s,t,u\}]$ in $\pi_0(\mathcal{D}_{A_3})$ is the non-trivial extension by the $\mathbb{Z}_2$ summand with index given by the corresponding class of vertices $[\{s,u\}]$ in $\pi_0(\mathcal{D}_{\bullet\bullet})$. Let $I=\pi_0(\mathcal{D}_{\bullet\bullet})$ and $J=\pi_0(\mathcal{D}_{A_3})$ then the matrix X is precisely X_W from Definition 2.5.52. $\square$

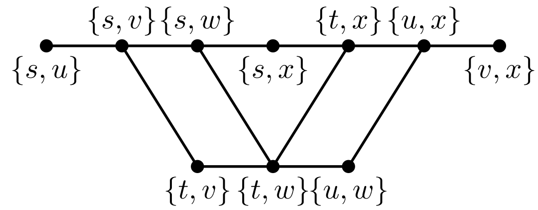
EXAMPLE 2.5.54. For example consider the Coxeter group defined by the following diagram:



then the diagram $\mathcal{D}_{A_3}$ is given by



and the diagram $\mathcal{D}_{\bullet\bullet}$ is given by



where the vertices corresponding to the two $W(A_3)$ subgroups: $\{s,u\}$ corresponding to $\{s,t,u\}$ and $\{v,x\}$ corresponding to $\{v,w,x\}$ are present at either end of $\mathcal{D}_{\bullet\bullet}$. The extension sequence takes the form

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] & B/A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & B \arrow[r] & H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & B \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

and we know from Lemma 2.5.51 that given the spectral sequence for the $W(A_3)$ subgroup corresponding to the representative for either of the $\mathbb{Z}_2$ components on the right, there is a non-trivial extension of this $\mathbb{Z}_2$ by the left $\mathbb{Z}_2$ to get a $\mathbb{Z}_4$. The extension matrix is therefore

$$X_W = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

with the row corresponding to the component of $\pi_0(\mathcal{D}_{\bullet\bullet})$ represented by $\{s,u\} = \{v,x\}$ and the columns to the two components of $\pi_0(\mathcal{D}_{A_3})$ represented by $\{s,t,u\}$ and $\{v,w,x\}$. In reality this can be realised as $B = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ with maps as follows.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & 0 \\
& a \arrow[r, mapsto] & (0,2a) & & \\
& & (b,c) \arrow[r, mapsto] & (b+c,b). &
\end{tikzcd}$$

Lemma 2.5.55. *There exist no non-trivial extensions from the*

$$\oplus\Big(\bigoplus_{\substack{W(H_3)\subseteq W\\ W(B_3)\subseteq W}} \mathbb{Z}_2\Big)$$

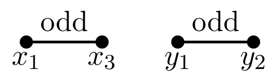
component of C to B.

Proof. We recall that subdiagrams of the form H_3 and B_3 in $\mathcal{D}_W$ represent these summands of C. We compute the spectral sequence for the groups corresponding to these diagrams, and compare to the third homology of the corresponding group $W(H_3)$ or $W(B_3)$ as computed using the De Concini - Salvetti resolution for finite Coxeter groups in [**18**]. Through these comparisons we observe that there are no non-trivial extensions present, as in the proof of Lemma 2.5.50. These calculations are found in Appendix B.

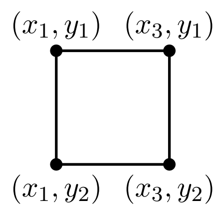
□

Lemma 2.5.56. *A class $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$ in C exists only when the spectral sequence is calculated for a Coxeter group W for which $\mathcal{D}_W$ has a subdiagram of the form $Y \sqcup A_1$ where Y is a 1-cycle in the Coxeter diagram $\mathcal{D}_{odd}$. That is a class in $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$ is represented by a loop containing only odd edges, along with a vertex disjoint from this loop, in $\mathcal{D}_W$.*

Proof. Let the vertices of the cycle be given by $\{t_1,\ldots,t_k\}$ and the disjoint vertex be given by s. Then the cycle given by $\{(t_1,s),\ldots(t_k,s)\}$ represents a cycle in $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$. To show that all classes in $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$ are represented by cycles of this form, suppose that $\{(x_1,y_1),\ldots,(x_p,y_p)\}$ represents a cycle. Without loss of generality, suppose $x_1 = x_2$. Then there exists an edge between (x_1,y_1) and (x_1,y_2) in $\mathcal{D}_{\bullet\bullet}$. That is, $m(y_1,y_2)$ must be odd. Now either $x_1 = x_3$ or $y_2 = y_3$, suppose $y_2 = y_3$ then it follows that $m(x_1,x_3)$ is odd. Then the vertices have the following form in the Coxeter diagram



and so in the diagram $\mathcal{D}_{\bullet\bullet}$ we have



and since this is a square, in the diagram $\mathcal{D}_{\bullet\bullet}^{\square}$ it is filled in, and thus the cycle $\{(x_1,y_1),(x_1,y_2),(x_3,y_2),(x_3,y_1)\}$ is a boundary. It follows that the sub-cycle $\{(x_1,y_1),(x_1,y_2),(x_3,y_2)\}$ of $\{(x_1,y_1),\ldots,(x_p,y_p)\}$ can be replaced with the vertex $\{(x_3,y_1)\}$, i.e. in $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$ the cycle $\{(x_1,y_1),\ldots,(x_p,y_p)\}$ is equal to the cycle $\{(x_3,y_1),(x_4,y_4)\ldots,(x_p,y_p)\}$. Without loss of generality, we can now assume that $x_3 = x_4$ and we are back to the start of the analysis of the cycle. Therefore, by reiterating this procedure we build a cycle equivalent, via boundaries, to $\{(x_1,y_1),\ldots,(x_k,y_k)\}$ and where $x_1 = x_i$ for all i. This is exactly a subdiagram of the form $Y \sqcup A_1$ in the Coxeter diagram $\mathcal{D}_W$, where Y is a loop in $\mathcal{D}_{odd}$. $\square$

Corollary 2.5.57. *There exists a possible extension problem between the $H_1(\mathcal{D}_{\bullet\bullet}^{\square};\mathbb{Z}_2)$ component in C and B, only when the spectral sequence is calculated for a Coxeter group W for which $\mathcal{D}_W$ has a subdiagram of the form $Y \sqcup A_1$ where Y is a 1-cycle in the Coxeter diagram $\mathcal{D}_{odd}$.*

2.5.58. Proof of Theorem B.

Theorem 2.5.59. *Given a finite rank Coxeter group W such that $\mathcal{D}_W$ does not have a subdiagram of the form $Y \sqcup A_1$, where Y is a loop in the Coxeter diagram $\mathcal{D}_{odd}$, there is an* *isomorphism*

$$\begin{aligned} H_3(W;\mathbb{Z}) \quad &\cong \quad H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \\ &\quad \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \oplus (H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) \end{aligned}$$

where each diagram is as described in Definition 2.1.12, and viewed as a simplicial complex. In this equation, $\bigcirc$ denotes the non-trivial extension of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ given by the extension matrix X_W defined in Definition 2.5.52.

If W is such that $\mathcal{D}_W$ has a subdiagram of the form $Y \sqcup A_1$ where Y is a 1-cycle in the Coxeter diagram $\mathcal{D}_{odd}$, then there is an isomorphism modulo extensions

$$\begin{aligned} H_3(W;\mathbb{Z}) \quad &\cong \quad H_0(\mathcal{D}_{odd};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \oplus H_0(\mathcal{D}_{\bullet\ \overset{even}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \\ &\quad \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \oplus (H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) \\ &\quad \oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2), \end{aligned}$$

where the unknown extensions involve the $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ summand.

PROOF. The two cases, when $\mathcal{D}_W$ contains a diagram of the form $Y \sqcup A_1$, and when it does not, are a direct result of Lemma 2.5.56 and Corollary 2.5.57. That is, if there is not a subdiagram of type $Y \sqcup A_1$ then the summand $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ is zero, and so there are no possible non-trivial extensions.

The other possible extension problems are solved in Lemmas 2.5.50, 2.5.51 and 2.5.55. This gives that the only non-trivial extension is the non-trivial extension of $H_0(\mathcal{D}_{A_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$, which is given by the extension matrix X_W of Definition 2.5.52 by Lemma 2.5.53.

The computation of the blue diagonal of the isotropy spectral sequence for the Davis complex at E^∞, alongside the solutions to these extension problems, gives the formula for $H_3(W;\mathbb{Z})$ as stated in the theorem. $\square$

CHAPTER 3

Background: Artin groups

3.1. Definition and examples

Recall from Definition 1.1.6 the definition of $\pi(a,b;k)$:

$$\pi(a,b;k) = \overbrace{aba\ldots}^{\text{length k}}$$

and let us refer to this as an *alternating product relation of length k*. Recall from Remark 1.1.7 the alternative presentation of a Coxeter group W with generating set S:

$$W = \left\langle S \,\middle|\, \begin{array}{cc} (s)^2 = e & \forall s \in S \\ \pi(s,t;m(s,t)) = \pi(t,s;m(s,t)) & \forall s,t \in S \end{array} \right\rangle.$$

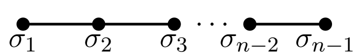
Then the corresponding Artin group is given by forgetting the involution condition.

Definition 3.1.1. For every Coxeter group W there is a corresponding Artin group A_W with presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \,|\, \forall s,t \in S, \pi(\sigma_s,\sigma_t;m(s,t)) = \pi(\sigma_t,\sigma_s;m(s,t))\rangle.$$

We note that the Coxeter diagram $\mathcal{D}_W$ also contains all the information about the Artin group presentation. Since this definition no longer implies that the generators are involutions, the group includes formal inverses σ_s^{-1} for each generator. Words in A are therefore strings of 'letters' for which the alphabet consists of σ_s and σ_s^{-1} for s in S.

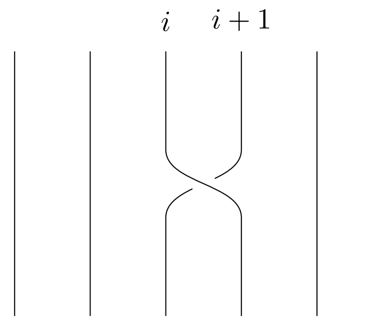
Example 3.1.2. The Artin group A_W corresponding to the Coxeter group $W = S_n$ is the *braid group*. We denote this $\mathcal{B}_n$. The corresponding diagram $\mathcal{D}_W$ is



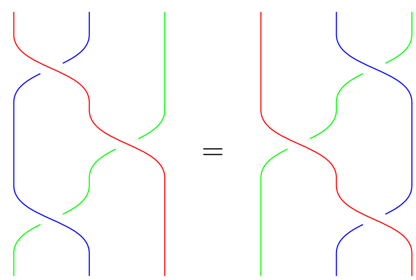
where we relabel σ_{s_i} to σ_i for ease of notation. From this diagram we see that there is no edge between generators when the subscript differs by 2 or more, and so these generators commute. When the subscript of two generators differs by 1 there is an unlabelled edge between them, which means that they satisfy an alternating product relation of length 3 on both sides. The presentation is therefore given by

$$\mathcal{B}_n = \left\langle \sigma_i \text{ for } s_i \in S \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \forall |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & 1 \leq i \leq (n-2) \end{array} \right\rangle$$

and this is the standard presentation for the braid group on n strands, with the generator σ_i given pictorially as



Here we note that if two generators have subscripts that differ by at least 2 they will involve disjoint strands, and so will commute. The relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ follows from the pictorial representation below



$$\sigma_i\sigma_{i+1}\sigma_i \quad = \quad \sigma_{i+1}\sigma_i\sigma_{i+1}$$

We call the half twists relating to the generators σ_i *positive twists* and those relating to generators σ_i^{-1} *negative twists.*

EXAMPLE 3.1.3. When all possible edges in the Coxeter diagram $\mathcal{D}_W$ are present and labelled with ∞ this corresponds to the Artin group A_W being the free group on $|S|$ generators. Recall an edge between two vertices s and t being labelled with ∞ corresponds to $m(s,t) = \infty$, or when viewed under the Artin presentation, there is no alternating product relation between σ_s and σ_t. Therefore the group has presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \rangle,$$

which is precisely the free group on $|S|$ generators.

EXAMPLE 3.1.4. When there are no edges in the Coxeter diagram $\mathcal{D}_W$ this corresponds to the Artin group A_W being the free abelian group on $|S|$ generators. Recall that no edge between two vertices s and t corresponds to $m(s,t) = 2$, or when viewed under the Artin presentation, there is an alternating product relation between σ_s and σ_t of length 2: σ_s and

σ_t commute. Therefore the group has presentation

$$A_W = \langle \sigma_s \text{ for } s \in S \mid \sigma_s\sigma_t = \sigma_t\sigma_s \, \forall s \neq t \in S \rangle,$$

which is precisely the free abelian group on $|S|$ generators.

Definition 3.1.5. When all of the edges in the Coxeter diagram are labelled with ∞, but not necessarily all possible edges are present (some $m(s,t)$ may be equal to 2) then the corresponding Artin group is called a *right angled* Artin group, or RAAG.

Definition 3.1.6. When the Coxeter group W is finite, i.e. when its diagram $\mathcal{D}_W$ is a disjoint union of diagrams from Proposition 1.1.12, then the corresponding Artin group A_W is called a *finite type* Artin group, or a *spherical* Artin group. Note that an Artin group itself is never finite, as all generators have infinite order.

Much of the known theory of Artin groups is concentrated around RAAGs and finite type Artin groups, though we do not restrict ourselves to either of these families in our results. In general little is known about Artin groups. For instance the following properties hold for finite type Artin groups [**13**]:

- there exists a finite model for the classifying space $K(A_W, 1)$,
- A_W is torsion free,
- the centre of A_W is $\mathbb{Z}$,
- A_W has solvable word and conjugacy problem

and to this date these properties are not known for general Artin groups. For instance the word problem requires an algorithm to determine if a word in the Artin group A_W, is equivalent via the group relations to the identity, or equivalently the conjugacy problem requires an algorithm to determine whether, given two words in A_W, one is a conjugate of the other. We now consider the first point in detail.

3.2. The $K(\pi, 1)$ conjecture

Definition 3.2.1. Given a CW complex X and a discrete group G we say that X is $K(G, 1)$, space if X is aspherical with fundamental group G. Such a space is a model for the *classifying space* BG of the group G, from which one can construct a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and hence calculate the (co)homology of G.

Example 3.2.2. We now look in detail at a $K(\mathcal{B}_n, 1)$ space for the braid group on n strands. It is known that the space of *unordered* configurations of n points in the plane is a classifying space for the braid group $\mathcal{B}_n$ (this was proved by Fox and Neuwirth [**24**]). An *ordered* configuration can be viewed as n ordered points on the complex plane $\mathbb{C}$, or one point in $\mathbb{C}^n$, such that no two of its co-ordinates are equal. The set in $\mathbb{C}^n$ consisting of points with two equal co-ordinates:

$$H_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j\}$$

is a codimension 1 subspace of $\mathbb{C}^n$, or a complex *hyperplane* of $\mathbb{C}^n$. Therefore the space of ordered configurations in $\mathbb{C}$ can be viewed as the following space:

$$\mathcal{M} = \mathbb{C}^n \backslash (\bigcup_{\substack{i,j \in \{1,\dots,n\} \\ i \neq j}} H_{i,j}).$$

Since we wish to consider unordered configurations, we take the quotient of $\mathcal{M}$ under the action of the symmetric group S_n which permutes the coordinates.

$$\mathcal{N} = \mathcal{M}/S_n$$

Putting this all together gives that the classifying space for the braid group $\mathcal{B}_n$ is equivalent to a hyperplane complement in $\mathbb{C}^n$, modulo the action of the symmetric group S_n. We note here that the symmetric group S_n is the Coxeter group associated to the braid group $\mathcal{B}_n$.

In general, one can associate a hyperplane complement to each Coxeter group W, such that there is a free action of the Coxeter group W. When you consider this hyperplane complement modulo this W action, the corresponding quotient has as its fundamental group the Artin group A_W. In some known cases this quotient space is a $K(A_W, 1)$, and this is conjectured to be true for all Artin groups. In the following section we make this precise, following Davis [**17**], notes by Paris [**40**] and the introduction to a paper on RAAGs by Charney [**13**].

Definition 3.2.3 (see Davis [**17**, 6.1.1]). A *linear reflection* on a vector space V is a linear transformation $r : V \to V$ such that r has order two and the fixed subspace of r is a hyperplane H_r in V. We call a group generated by such linear reflections a *reflection group.*

Proposition 3.2.4 (see Davis [**17**, 6.6.3]). *If W is a finite group generated by a set of linear reflections S on a finite dimensional vector space V then (W, S) is a Coxeter system.*

We can associate to such a group W a bilinear form B on V which encodes the information of each generating reflection, see for example [**13**] or [**17**, Chapter 6]. When the reflection group is finite, B is positive definite and so defines an inner product on V. Identifying (V, B) and $(\mathbb{R}^n, \cdot)$ identifies the reflection hyperplanes of W in V with a finite hyperplane arrangement in $\mathbb{R}^n$:

$$\mathcal{A} = \{H_r \,|\, r \text{ is a reflection in W}\}.$$

It follows that every point in $\mathbb{R}^n$ with non-trivial stabiliser under the group action of W lies in a hyperplane in $\mathcal{A}$. Complexifying gives an arrangement of complex hyperplanes in $\mathbb{C}^n$ such that W acts freely on the complement:

$$M(\mathcal{A}) = \mathbb{C}^n \backslash (\bigcup_{H_r \in \mathcal{A}} \mathbb{C} H_r).$$

Artin groups were first introduced by Brieskorn [**9**] as the fundamental groups of the quotient $M(\mathcal{A})/W$ and in the 1970s Deligne proved the following theorem [**19**].

THEOREM 3.2.5 (Deligne's Theorem, see Charney [**13**, 1.1]). *For W a finite Coxeter group and A_W the associated Artin group, $M(\mathcal{A})/W$ is aspherical with fundamental group A_W, that is $M(\mathcal{A})/W$ is a $K(A_W, 1)$.*

For arbitrary Artin groups, there is a well known conjecture called the $K(\pi, 1)$ *conjecture*, formulated by Arnol'd, Thom and Pham. This conjecture states than an analogue of Deligne's theorem holds for all Artin groups. The analogue of the hyperplane complement was formulated by Vinberg, and is as follows.

DEFINITION 3.2.6 (see Davis [**17**, A.1.8]). A *convex polyhedral cone* in a finite vector space V is the intersection of a finite set of linear half-spaces in V.

DEFINITION 3.2.7 (see Paris [**40**]). Let V be a finite-dimensional real vector space and let $\overline{C}_0$ be a closed convex polyhedral cone in V with non-empty interior denoted C_0. Define a *wall* of $\overline{C}_0$ to be a hyperplane of V determined by a codimension 1 face of $\overline{C}_0$. Let $H_1, \dots H_n$ be the walls of $\overline{C}_0$ and let s_i be a linear reflection which fixes H_i. Denote W to be the subgroup of $GL(V)$ generated by $S = \{s_1, \dots s_n\}$.

DEFINITION 3.2.8. With notation as above, W and S describe a *Vinberg system* (W, S), if for all w in $W\backslash\{1\}$ the transformation of C_0 under w is disjoint from C_0, i.e. $wC_0 \cap C_0 = \emptyset$.

DEFINITION 3.2.9. Given a Vinberg system (W, S) let

$$\bar{I} = \bigcup_{w\in W} w\overline{C}_0.$$

Then the interior I is called the *Tits cone* of the system.

The following theorem of Vinberg is a prominent result linking Coxeter groups and hyperplane arrangements.

THEOREM 3.2.10 (Vinberg, see Paris [**40**, 1.1]). *With the above notation, let (W, S) be a Vinberg system. Then the following are true:*

(1) W is a Coxeter group with generating set S.
(2) $\bar{I}$ is a convex cone and I is non-empty.
(3) The Tits cone I is invariant under the action of W, and W acts properly and discontinuously on I.
(4) If $x \in I$ satisfies that the stabiliser of x is non-trivial, then there exists a reflection r in W such that $r(x) = x$.

DEFINITION 3.2.11. For a Vinberg system (W, S) we denote by $\mathcal{R}$ the set of reflections in W, as before we set $\mathcal{A} = \{H_r \,|\, r \in \mathcal{R}\}$. Then from the previous theorem $\mathcal{A}$ is a hyperplane arrangement in I. We set

$$\mathcal{M}(\mathcal{A}) = (I \times I)\backslash(\bigcup_{H\in\mathcal{A}} H \times H).$$

This agrees with our definition of $M(\mathcal{A})$ when $I = V$, giving $\mathcal{A}$ finite. By the previous theorem, W acts freely and properly discontinuously on $\mathcal{M}(\mathcal{A})$ and hence we can take the quotient

$$\mathcal{N}(\mathcal{A}) = \mathcal{M}(\mathcal{A})/W.$$

THEOREM 3.2.12 (Van der Lek, see Paris [**40**, 1.2]). *Let (W, S) be a Vinberg system and $\mathcal{N}(\mathcal{A})$ be defined as above. Then the fundamental group of $\mathcal{N}(\mathcal{A})$ is isomorphic to the associated Artin group A_W.*

This result led to the formulation of Deligne's theorem as a conjecture in this set up:

CONJECTURE 3.2.13. *The space $\mathcal{N}(A_W)$ is a $K(A_W, 1)$ space.*

REMARK 3.2.14. It is worth noting here a reformulation of the conjecture in terms of a finite dimensional CW-complex called the *Salvetti complex*, denoted by $Sal(\mathcal{A})$ and introduced by Salvetti in [**43**], for a hyperplane arrangement $\mathcal{A}$ in a finite dimensional real vector space V. The Salvetti complex is defined in terms of cosets, much like the Davis complex from section 1.3. Paris extends this definition to any infinite hyperplane arrangement in a non-empty convex cone I [**40**] and proves that $Sal(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$ have the same homotopy type. The $K(\pi, 1)$ conjecture can therefore be restated as a conjecture about the Salvetti complex.

In general, Charney and Davis [**14**] proved the following.

THEOREM 3.2.15 (Charney and Davis [**14**]). *For (W, S) a Vinberg system, the homotopy type of the corresponding $\mathcal{M}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$ depends only on the Coxeter diagram $\mathcal{D}_W$ of the associated Coxeter group W.*

The $K(\pi, 1)$ conjecture has been proved for large classes of Artin groups [**40**]. For example the conjecture holds for:

- *Finite type* Coxeter groups: this is Deligne's Theorem 3.2.5.
- *Large* type Coxeter groups: when the Coxeter group has relations $m(s, t) \geq 3$ for all $s \neq t$.
- Coxeter groups of *dimension 2*: when all T in $\mathcal{S}$ satisfy $|T| \leq 2$.
- Coxeter groups of *FC type*: when $\mathcal{S}{=}\mathcal{S}^{<\infty} := \{T \subseteq S \,|\, m(s, t) \neq \infty \,\forall s, t \in T\}$.

However the conjecture has not been proved to hold for general Artin groups to date. We apply a reformulation of the $K(\pi, 1)$ conjecture to our results, which involves the Artin monoid A^+ discussed in the next chapter.

CHAPTER 4

Background: Artin monoids

This section follows Jean Michel *A note on words in braid monoids* [**38**] and Brieskorn and Saito *Artin-Gruppen und Coxeter-Gruppen* [**11**].

4.1. Definition and examples

Definition 4.1.1. The *Artin monoid* A_W^+ of an Artin group A_W associated to a Coxeter group W is defined as the monoid with the same presentation as A:

$$A_W^+ = \langle \sigma_s \text{ for } s \in S \,|\, \forall \sigma_s, \sigma_t, \pi(\sigma_s, \sigma_t; m(s,t)) = \pi(\sigma_t, \sigma_s; m(s,t)) \rangle^+.$$

Words in A^+ are therefore strings of 'letters' for which the alphabet consists of σ_s for s in S.

Remark 4.1.2. The group completion of A_W^+ is A_W. That is there is a monoid homomorphism A_W^+ to A_W (in this case given by inclusion), and A_W satisfies the universal property that any monoid homomorphism from A_W^+ to a group G will factor uniquely through A_W.

Example 4.1.3. The braid monoid $\mathcal{B}_n^+$ is the monoid associated to the Artin group $\mathcal{B}_n$, the braid group, and Coxeter group S_n, the symmetric group. The braid monoid consists of words in the braid group made from the positive generators σ_i. In terms of the braid diagrams these can be viewed as braids consisting of only positive twists.

Definition 4.1.4. We define a *submonoid* M^+ of an Artin monoid A^+ to be a *parabolic submonoid* if the monoid M^+ is generated by the set $M^+ \cap S$. We call this generating set for the monoid S_M.

In this thesis, when referring to an submonoid of an Artin monoid, we will always mean a parabolic submonoid.

4.2. Divisors in Artin monoids: general theory

Throughout this section let A^+ be an Artin monoid.

Definition 4.2.1. Define the *length function* on an Artin monoid A^+ corresponding to a Coxeter system (W, S)

$$\ell : A^+ \to \mathbb{N}$$

to be the function which maps α in A^+ to the minimum word length required to express α in terms of the generators, as in the definition for Coxeter groups.

REMARK 4.2.2. Note here that since there are no inverses in Artin monoids, multiplication of two words does not lead to any cancellation, and therefore multiplication corresponds to addition of lengths, i.e. $\ell(ab) = \ell(a) + \ell(b)$.

DEFINITION 4.2.3. For words α and β in A^+ we say that $\alpha \preceq_R \beta$ if for some γ in A^+ we have $\beta = \gamma\alpha$, that is α appears on the right of some expression for β. We say that β is *right-divisible* by α, or alternatively that α *right divides* β.

We define $\preceq_L$ similarly, though we do not use this definition in this thesis.

PROPOSITION 4.2.4 (see Michel [**38**, Prop 2.4]). *Artin monoids satisfy left and right cancellation, i.e. for a, b and c in A^+,*

$$ab = ac \Rightarrow b = c$$
$$ba = ca \Rightarrow b = c.$$

We now consider work by Brieskorn and Saito in their 1972 paper *Artin-Gruppen und Coxeter-Gruppen* [**11**]. They consider least common multiples and greatest common divisors of sets of words in the Artin monoid. We are interested in the notion of least common multiple.

DEFINITION 4.2.5. Given a set of elements $\{g_j\}_{j\in J}$ in an Artin monoid A^+, a *common multiple* β is an element in A^+ which is right divisible by all g_j. That is $g_j \preceq_R \beta$ for all g_j in the set. A *least common multiple* is a common multiple which right-divides all other common multiples.

PROPOSITION 4.2.6 (Brieskorn and Saito [**11**, 4.1]). *A finite set of elements in an Artin monoid either has a least common multiple or no common multiple at all.*

LEMMA 4.2.7 (Brieskorn and Saito [**11**]). *The letters arising in a least common multiple of a set of words in an Artin monoid are only those letters which appear in the words themselves.*

DEFINITION 4.2.8. Let E be a set of words in the Artin monoid A^+. Denote the least common multiple (if it exists) of E by $\Delta(E)$. For α and β two words in A^+ denote the least common multiple of α and β (if it exists) by $\Delta(\alpha, \beta)$.

DEFINITION 4.2.9. Consider a submonoid M^+ of an Artin monoid A^+. Denote the generating set for the Coxeter group associated to the monoid by S and to the submonoid by S_M. Given a word α in A^+ we define two *end sets* for the word.

$$\begin{aligned} EndGen_M(\alpha) &= \{\sigma_s \mid s \in S_M, \sigma_s \preceq_R \alpha\} \\ EndMon_M(\alpha) &= \{\beta \in M \mid \beta \preceq_R \alpha\} \end{aligned}$$

REMARK 4.2.10. $EndGen_M(\alpha)$ is exactly the letters σ_s for s in S_M that the word α can end with, and $EndMon_M(\alpha)$ is exactly the words in M^+ that the word α can end with. Note that $EndGen_M(\alpha)$ is a subset of $EndMon_M(\alpha)$, consisting of words that have length 1 and that if $EndMon_M(\alpha) = \emptyset$ then this implies that α has no right-divisors in M^+.

4.3. Divisors in Artin monoids: theory for proof

Much of the proof in Chapter 6 is concerned with algebraic manipulation of words in the Artin monoid. Here we introduce some technical definitions and lemmas used in the proof. We build up a theory of monoid cosets in the case of Artin monoids, noting that this is very particular to Artin monoids and many of these techniques will not work with general monoids. The theory developed in this section is new unless cited.

LEMMA 4.3.1. *Given α in A^+, and M^+ a submonoid of A^+, the set $EndMon_M(\alpha)$ has a least common multiple $\Delta(EndMon_M(\alpha)) = \beta$ which lies in the submonoid M^+. That is there exists β in M^+ and γ in A^+ such that $\alpha = \gamma\beta$, and if β' in M^+ and γ' in A^+ satisfy $\alpha = \gamma'\beta'$, it follows that $\beta' \preceq_R \beta$.*

PROOF. From Proposition 4.2.6 we know that if a common multiple exists, then $\Delta(EndMon_M(\alpha))$ exists. α itself is a common multiple of all elements in $EndMon_M(\alpha)$, by definition of $EndMon_M(\alpha)$. Furthermore Lemma 4.2.7 notes that only letters appearing in $EndMon_M(\alpha)$ will appear in $\Delta(EndMon_M(\alpha))$. By definition, these are letters in M^+ and so $\Delta(EndMon_M(\alpha))$ lies in M^+. The second part of the lemma applies the definition of the least common multiple. $\square$

REMARK 4.3.2. For a word α in A^+ let the least common multiple of $EndMon_M(\alpha)$ be β. We write $\overline{\alpha}$ *with respect to* M^+ for the word $\overline{\alpha}$ in A^+ such that $\alpha = \overline{\alpha}\beta$. It will always be clear in the text for which submonoid M^+ we are taking the reduction with respect to.

DEFINITION 4.3.3. For A^+ an Artin monoid and M^+ a submonoid, let $A^+(M)$ be the following set

$$A^+(M) = \{\overline{\alpha} \text{ with respect to } M^+ \mid \alpha \in A^+\}.$$

That is, $A^+(M)$ is the set of words in A^+ which do not end in any word from M.

LEMMA 4.3.4. *For all α in A^+ and all β in M^+, $\overline{\alpha} = \overline{\alpha\beta}$ where the reduction is taken with respect to M^+.*

PROOF. Let $\overline{\alpha} = \gamma$, so $\alpha = \gamma\eta$ for some η in M^+, and $EndMon_M(\gamma) = \emptyset$ i.e. γ has no right divisors in M^+. Then $\alpha\beta = \gamma\eta\beta$ and since η and β are both in M^+, it follows that $\eta\beta \in EndMon_M(\alpha\beta)$. If $\eta\beta$ is the least common multiple of $EndMon_M(\alpha\beta)$ then it follows that $\overline{\alpha\beta} = \gamma = \overline{\alpha}$ so we are done. Therefore suppose that $\eta\beta$ is not the least common multiple of $EndMon_M(\alpha\beta)$, and note that $\eta\beta$ will be a right divisor of the actual least common multiple. Then there exists some ζ in M^+ of length at least 1 such that $\zeta\eta\beta$ is the least common multiple of $EndMon_M(\alpha\beta)$. It follows that there exists a $\gamma' = \overline{\alpha\beta}$ with $EndMon_M(\gamma') = \emptyset$ and $\alpha\beta = \gamma'\zeta\eta\beta$. But $\alpha\beta = \gamma\eta\beta$ and it follows from cancellation that $\gamma = \gamma'\zeta$. Since ζ is in M^+ with length at least 1 it follows that $\zeta \in EndMon_M(\gamma)$ which contradicts $EndMon_M(\gamma) = \emptyset$. Therefore $\eta\beta$ is the least common multiple of $EndMon_M(\alpha\beta)$ and it follows that $\overline{\alpha\beta} = \gamma = \overline{\alpha}$. $\square$

Definition 4.3.5. Consider now the relation $\sim$ on A^+ given by

$$\alpha_1 \sim \alpha_2 \iff \alpha_1\beta_1 = \alpha_2\beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } M^+$$

where M^+ is a submonoid of M. Again, if we are using this relation it will be made clear which submonoid M^+ is being considered. We have that $\sim$ is symmetric and reflexive. Let $\approx$ be the transitive closure of $\sim$. That is, $\alpha_1 \approx \alpha_2$ if there is a chain of elements in A^+:

$$\alpha_1 \sim \tau_1 \sim \tau_2 \sim \ldots \sim \tau_k \sim \alpha_2$$

for some k. Denote the equivalence class of α in A^+ under the relation $\approx$ with respect to the submonoid M^+ as $[\alpha]_M$.

Lemma 4.3.6. *The equivalence classes under $\approx$ with respect to the submonoid M^+ are in one to one correspondence with the set $A^+(M)$, that is for all α_1 and α_2 in A^+:*

$$[\alpha_1]_M = [\alpha_2]_M \iff \overline{\alpha_1} = \overline{\alpha_2}$$

Proof. ($\Rightarrow$) If $\overline{\alpha_1} = \overline{\alpha_2} = \gamma$ with respect to M^+, then $\alpha_1 \sim \gamma \sim \alpha_2$ so it follows $\alpha_1 \approx \alpha_2$. We need to show that if $\alpha_1 \approx \alpha_2$ then $\overline{\alpha_1} = \overline{\alpha_2}$. Since $\alpha_1 \approx \alpha_2$ there is a chain $\alpha_1 \sim \tau_1 \sim \tau_2 \sim \ldots \sim \tau_k \sim \alpha_2$, so if we show that $\overline{\eta} = \overline{\zeta}$ whenever $\eta \sim \zeta$ for η and ζ in A^+ it will follow that $\overline{\alpha}_1 = \overline{\tau}_1 = \overline{\tau}_2 = \ldots = \overline{\tau}_k = \overline{\alpha}_2$. Since $\eta \sim \zeta$ it follows that for some β_1 and β_2 in M^+, $\eta\beta_1 = \zeta\beta_2$. From Lemma 4.3.4 we know that $\overline{\eta} = \overline{\eta\beta_1}$ and similarly $\overline{\zeta} = \overline{\zeta\beta_2}$ so it follows

$$\overline{\eta} = \overline{\eta\beta_1} = \overline{\zeta\beta_2} = \overline{\zeta}$$

which completes the proof. $\square$

Proposition 4.3.7. *For M^+ a submonoid of A^+, $A^+ \cong A^+(M) \times M^+$ as sets, via the bijection*

$$\begin{aligned} p : A^+ &\rightarrow A^+(M) \times M^+ \\ \alpha &\mapsto (\overline{\alpha}, \beta) \textit{ where } \alpha = \overline{\alpha}\beta \end{aligned}$$

and this decomposition respects the right action of M^+ on A^+.

Proof. To show p is surjective: consider $(\gamma, \beta) \in A^+(M) \times M^+$. Due to Lemma 4.3.4 for any $\beta \in M^+$ we have $\overline{\alpha\beta} = \overline{\alpha}$. Therefore $\gamma\beta$ satisfies $p(\gamma\beta) = (\gamma, \beta)$ since $\overline{\gamma\beta} = \overline{\gamma} = \gamma$ (we have $\gamma \in A^+(M)$ so $EndMon_p(\gamma) = \emptyset$). To show injectivity, suppose $p(\alpha_1) = p(\alpha_2)$, that is $(\overline{\alpha_1}, \beta_1) = (\overline{\alpha_2}, \beta_2)$. This translates to $\alpha_1 = \overline{\alpha_1}\beta_1 = \overline{\alpha_2}\beta_2 = \alpha_2$, therefore p is injective. Under this decomposition, the action of m in M^+ satisfies $p(\alpha \cdot m) = (\overline{\alpha}, \beta \cdot m)$ where $\alpha = \overline{\alpha}\beta$, again due to Lemma 4.3.4. Therefore the right action of M^+ under this decomposition acts trivially on the first factor and as right multiplication on the second. $\square$

Proposition 4.3.8 (see Michel [**38**, 1.5]). *If generators s and t in S_M are in $EndGen_M(\alpha)$ for some α in A^+ then $\Delta(s,t)$ is in $EndMon_M(\alpha)$.*

LEMMA 4.3.9. *Consider F a subset of $EndMon_M(\alpha)$ for some submonoid M^+ of A^+ and some α in A^+. Then $\Delta(F)$ is in $EndMon_M(\alpha)$.*

PROOF. Since F is a subset of $EndMon_M(\alpha)$, which has a least common multiple, then certainly $\Delta(F)$ exists. The element $\Delta(F)$ divides all other common multiples of F. Since $\Delta(EndMon_M(\alpha))$ is a common multiple for $EndMon_M(\alpha)$, it is certainly a common multiple for F. Therefore $\Delta(F) \preceq_R \Delta(EndMon_M(\alpha))$ and it follows that $\Delta(F)$ is in $EndMon_M(\alpha)$. □

DEFINITION 4.3.10. Words α and β in an Artin monoid are defined to *letterwise commute* if each letter in the word α commutes with every letter in the word β, and the set of letters that α contains is disjoint from the set of letters that β contains.

LEMMA 4.3.11. *If β and γ are in $EndMon_M(\alpha)$ and β and γ letterwise commute, it follows that:*

- $\Delta(\beta,\gamma) = \beta\gamma = \gamma\beta$
- $\Delta(\beta,\gamma)$ *is in* $EndMon_M(\alpha)$

PROOF. Since β and γ letterwise commute, they contain distinct generators. The relations in any Artin monoid have the same generators on both sides of the equality, therefore every letter in the words β and γ must appear in $\Delta(\beta,\gamma)$. If both β and γ have length 1, say $\beta = \sigma$ and $\gamma = \tau$ for generators σ and τ then since the words letterwise commute it follows that σ commutes with τ. Therefore since $\sigma\tau = \tau\sigma$ and both generators must appear in $\Delta(\beta,\gamma)$ it follows that

$$\Delta(\beta,\gamma) = \sigma\tau = \tau\sigma = \beta\gamma = \gamma\beta$$

as required. Similarly, if $\beta = \sigma_1 \dots \sigma_k$ has length k, and $\gamma = \tau$ has length 1 then since the words contain distinct generators it follows:

$$\Delta(\beta,\tau) = \Delta(\sigma_1 \dots \sigma_k, \tau) = (\sigma_1 \dots \sigma_k)\tau = \tau(\sigma_1 \dots \sigma_k) = \beta\tau = \tau\beta.$$

Suppose now that $\beta = \sigma_1 \dots \sigma_k$ has length k and $\gamma = \tau_1 \dots \tau_l$ has length l. It is certainly true that $\beta \preceq_R \beta\gamma$ and $\gamma \preceq_R \beta\gamma$. We must show that if x in A^+ is a common multiple of β and γ then $\beta\gamma = \gamma\beta$ is in $EndMon_M(x)$. Since x is a common multiple it follows that

$$x = y\beta = y\sigma_1 \dots \sigma_k \qquad\qquad x = z\gamma = z\tau_1 \dots \tau_l$$

for some y and z in A^+, and since both β and τ_l are in $EndMon_M(x)$ we have from Lemma 4.3.9 that $\Delta(\beta,\tau_l) = \beta\tau_l = \tau_l\beta$ is in $EndMon_M(x)$. Therefore since $x = y\beta$, by cancellation of β, τ_l is in $EndMon_M(y)$ and so $y = y_1\tau_l$ for some y_1 in A^+. The previous equation becomes

$$x = x_1\tau_l = y\beta = y_1\tau_l\beta = y_1\beta\tau_l \qquad\qquad x = x_1\tau_l = z\gamma = z\tau_1 \dots \tau_l$$

for some x_1 in A^+. By cancellation of τ_l we have that $x_1 = y_1\beta$ and $x_1 = z\tau_1 \dots \tau_{l-1}$. Therefore x_1 satisfies both β and τ_{l-1} are in $EndMon_M(x_1)$ and, repeating the same argument, we conclude that τ_{l-1} is in $EndMon(y_1)$ and so $y_1 = y_2\tau_{l-1}$ for some y_2 in A^+. The previous equation becomes

$$x = x_2\tau_{l-1}\tau_l = y\beta = y_2\tau_{l-1}\tau_l\beta = y_2\beta\tau_{l-1}\tau_l$$

for some x_2 in A^+. Continuing in this fashion we arrive at

$$x = x_l\tau_1 \dots \tau_l = y\beta = y_l\tau_1 \dots \tau_l\beta = y_l\gamma\beta$$

for some x_l in A^+, and so $\beta\gamma = \gamma\beta$ is in $EndMon_M(x)$ as required. This shows that $\Delta(\beta, \gamma) = \beta\gamma = \gamma\beta$.

Invoking Lemma 4.3.9 with $F = \{\beta, \gamma\}$ we have $\Delta(F) = \Delta(\beta, \gamma) = \beta\gamma = \gamma\beta$ is in $EndMon_M(\alpha)$. □

LEMMA 4.3.12. *If words α, a and b in A^+, are such that $b \preceq_R \alpha a$ and a and b letterwise commute, then it follows that $b \preceq_R \alpha$.*

PROOF. An equivalent way of writing $m \preceq_R n$ for m, n in A^+ is $m \in EndMon_A(n)$ where the end set is taken with respect to the full monoid A^+. Since a and b are both in $EndMon_A(\alpha a)$ it follows that $\Delta(a, b)$ is in $EndMon_A(\alpha a)$, from Lemma 4.3.9. Since a and b letterwise commute, $\Delta(a, b) = ab = ba$. Therefore ba is in $EndMon_A(\alpha a)$, and by cancellation of a it follows that b is in $EndMon_A(\alpha)$ as required. □

4.4. Relation to the $K(\pi, 1)$ conjecture

In 2002 Dobrinskaya published a paper relating the classifying space of the Artin monoid BA_W^+ to the $K(\pi, 1)$ conjecture. This was later translated into English as *Configuration Spaces of Labelled Particles and Finite Eilenberg - MacLane Complexes* [**20**]. The main result of the paper was the following:

THEOREM 4.4.1 (Dobrinskaya [**20**, Theorem 6.3]). *Given an Artin group A_W and its associated monoid A_W^+, the $K(\pi, 1)$ conjecture holds if and only if the natural map between their classifying spaces, $BA_W^+ \to BA_W$, is a homotopy equivalence.*

She proved this via the introduction of a finite subset of the Artin monoid $A_f^+ \subset A^+$ and a notion of classifying space BA_f^+ for this subset, such that the map $BA_f^+ \to BA^+$ was a homotopy equivalence. She then proved that BA_f^+ was homotopy equivalent to the hyperplane complement $\mathcal{M}(\mathcal{A})$ defined in Definition 3.2.11. Putting this together gave that the classifying space for the monoid BA^+ was homotopy equivalent to $\mathcal{M}(\mathcal{A})$ [**20**, Theorem 6.2], which completes the proof.

4.5. Semi-simplicial constructions with monoids

4.5.1. Semi-simplicial background.

DEFINITION 4.5.2 (see Ebert and Randal-Williams [**21**, 1.1]). Let Δ denote the category which has as its objects the non-empty finite ordered sets $[n] = \{0, 1, \ldots, n\}$, and as its morphisms monotone functions. These functions are generated by the basic functions:

$$\begin{aligned} D^i : [n] &\rightarrow [n+1] \text{ for } 0 \leq i \leq n \\ \{0, 1, \ldots, n\} &\mapsto \{0, 1, \ldots, \widehat{i}, \ldots, n+1\} \\ S^i : [n+1] &\rightarrow [n] \text{ for } 0 \leq i \leq n \\ \{0, 1, \ldots, n+1\} &\mapsto \{0, 1, \ldots, i, i, \ldots n\} \end{aligned}$$

The opposite category Δ^{op} is known as the simplicial category. We denote the opposite of the maps D^i by ∂_i and the opposite of the maps S^i by s_i. We call these the *face maps* and the *degeneracy maps* respectively.

Let $\Delta_{inj} \subset \Delta$ be the subcategory of Δ which has the same objects but only the injective monotone maps as morphisms, generated by the D_i. The opposite category Δ_{inj}^{op} is known as the semi-simplicial category and its morphisms are therefore generated by the face maps ∂_i.

DEFINITION 4.5.3 (see Ebert and Randal-Williams [**21**, 1.1]). A *simplicial object* in a category $\mathcal{C}$ is a covariant functor $X_\bullet : \Delta^{op} \rightarrow \mathcal{C}$. A *semi-simplicial object* is a functor $X_\bullet : \Delta_{inj}^{op} \rightarrow \mathcal{C}$. We denote $X_\bullet([n])$ by X_n. A *(semi-)simplicial map* $f : X_\bullet \rightarrow Y_\bullet$ is a natural transformation of functors, and in particular has components $f_n : X_n \rightarrow Y_n$. Simplicial objects in $\mathcal{C}$ form a category denoted $s\mathcal{C}$, and semi-simplicial objects a category denoted $ss\mathcal{C}$. When $\mathcal{C}$ is equal to **Set** the (semi-)simplicial object is called a *(semi-)simplicial set*.

REMARK 4.5.4. A semi-simplicial object in a category $\mathcal{C}$ is equivalent to the following data:

(a) An object X_p in $\mathcal{C}$, for $p \geq 0$
(b) Morphisms $\partial_i^p : X_p \rightarrow X_{p-1}$ for $0 \leq i \leq p$ and all $p \geq 0$ in $\mathcal{C}$ called *face maps* which satisfy the following *simplicial identities*

$$\partial_i^{p-1} \partial_j^p = \partial_{j-1}^{p-1} \partial_i^p \text{ if } i < j.$$

DEFINITION 4.5.5 ([see Ebert and Randal-Williams [**21**, 1.3]). An augmented semi - simplicial object in $\mathcal{C}$ is a triple $(X_\bullet, X_{-1}, \epsilon_\bullet)$ such that $X_\bullet$ is a semi-simplicial object in $\mathcal{C}$, X_{-1} is an object of $\mathcal{C}$ and $\epsilon_\bullet$ is a family of morphisms such that $\epsilon_p : X_p \rightarrow X_{-1}$ and $\epsilon_{p-1} \circ \partial_i = \epsilon_p$ for all $p \geq 1$ and $0 \leq i \leq p$.

EXAMPLE 4.5.6 (see Ebert and Randal-Williams [**21**, 1.2]). The standard n-simplex has two equivalent manifestations: as a simplicial object in **Set** and as an object in **Top**. When viewed as a simplicial set the standard n-simplex is denoted $\Delta_\bullet^n$ and is defined via the functor $\Delta_m^n = \Delta_\bullet^n([m]) = hom_\Delta([m], [n])$ for all $[m]$ in Δ^{op}. When viewed as an object in **Top** the standard n-simplex is denoted $|\Delta^n|$ and defined to be

$$|\Delta^n| := \Big\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \forall i\Big\}.$$

One can associate to a morphism $\phi : [m] \to [n]$ in Δ a continuous map

$$\begin{aligned} \phi_* : |\Delta^n| \quad &\to \quad |\Delta^m| \\ (t_0, \dots, t_n) \quad &\mapsto \quad (s_0, \dots, s_m) \text{ where } s_j = \sum_{\phi(i)=j} t_i. \end{aligned}$$

That is, morphisms send the jth vertex of the simplex $|\Delta^n|$ to the $\phi(j)$th vertex of $|\Delta^m|$ and extend linearly. Under this viewpoint the map D^i_* sends $|\Delta^n|$ to the ith face of $|\Delta^{n+1}|$ and the map S^i_* collapses together the ith and $(i+1)$st vertices of $|\Delta^{n+1}|$ to give a map to $|\Delta^n|$.

Applying several face maps in a row can be denoted as a tuple $(\partial^{p-1}_{i_1}, \partial^{p-2}_{i_2}, \dots, \partial^{p-k}_{i_k})$ where $\partial^{p-1}_{i_1}$ is the first face map to be applied, followed by $\partial^{p-2}_{i_2}$, etc. For ease of notation we assume that the second map in the tuple maps from the target of the first map, and the third from the target of the second map etc., and so we dispense with superscripts, writing the tuple as $(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k})$.

Lemma 4.5.7. *With the above notation, the tuple of face maps can be organised such that $i_{j+1} \geq i_j$ for all j.*

Proof. Suppose $i_{j+1} < i_j$ in the tuple $(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_k})$.. The simplicial identities then tell us that applying ∂_{i_j} before $\partial_{i_{j+1}}$ is the same as applying $\partial_{i_{j+1}}$ before ∂_{i_j-1}, i.e.

$$\partial_{i_{j+1}}\partial_{i_j} = \partial_{i_j-1}\partial_{i_{j+1}} \text{ since } i_{j+1} < i_j$$

Therefore $(\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_j}, \partial_{i_{j+1}}, \dots, \partial_{i_k}) = (\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_{j+1}}, \partial_{i_j-1}, \dots, \partial_{i_k})$. Since $i_{j+1} < i_j$, it follows that $i_j - 1 \geq i_{j+1}$. Relabelling $i_j := i_{j+1}$ and $i_{j+1} := i_j - 1$ gives $(\partial_{i_1}, \partial_{i_2} \dots, \partial_{i_j}, \partial_{i_{j+1}}, \dots, \partial_{i_k})$ such that $i_{j+1} \geq i_j$. Applying this process reduces the sum $\sum_{j=1}^{k} i_j$ by one, and therefore iteration of this process must terminate. If we apply this process enough times, we get $i_{j+1} \geq i_j$ for all j. □

Definition 4.5.8 (see Ebert and Randal-Williams [**21**, 1.2]). The *geometric realisation* of a semi-simplicial space is denoted by $\|X_\bullet\|$ and defined to be

$$\|X_\bullet\| := \coprod_{n\geq 0} X_n \times |\Delta^n| / \sim$$

where $\sim$ is generated by $(x, t) \sim (y, u)$ whenever $\partial_i(x) = y$ and $D^i(u) = t$.

Definition 4.5.9. Given a semi-simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ there is an induced map $\|f_\bullet\| : \|X_\bullet\| \to \|Y_\bullet\|$ which we call the *geometric realisation of the semi-simplicial map $f_\bullet$*.

Definition 4.5.10 (see Ebert and Randal-Williams [**21**, 1.4]). A *bi-semi-simplicial* object in a category $\mathcal{C}$ is a functor $X_{\bullet\bullet} : (\Delta_{inj} \times \Delta_{inj})^{op} \to \mathcal{C}$. We write $X_{p,q} = X_{\bullet\bullet}([p],[q])$. We write the image of the standard face maps in each simplicial direction $(\partial_i \times id)$ and $(id \times \partial_j)$, as $\partial_{i,\bullet}$ and $\partial_{\bullet,j}$. We note that $(\partial_i \times \partial_j) = (\partial_{i,\bullet} \circ \partial_{\bullet,j}) = (\partial_{\bullet,j} \circ \partial_{i,\bullet}) : X_{p,q} \to X_{(p-1),(q-1)}$ and we denote this map $\partial_{i,j}$. When $\mathcal{C}$ is equal to **Set** the bi-semi-simplicial object is called a *bi-semi-simplicial set.*

Remark 4.5.11. A bi-semi-simplicial set can be viewed as a semi-simplicial object in ss**Set** in two ways:

1. $X_{\bullet,q} : [p] \mapsto (X_\bullet : [q] \mapsto X_{p,q})$ with face maps $\partial_{i,\bullet}$.
2. $X_{p,\bullet} : [q] \mapsto (X_\bullet : [p] \mapsto X_{p,q})$ with face maps $\partial_{\bullet,j}$.

Definition 4.5.12 (see Ebert and Randal-Williams [**21**, 1.2]). Given a bi-semi-simplicial set $X_{\bullet,\bullet}$ we define its *geometric realisation* to be

$$\|X_{\bullet,\bullet}\| = \coprod_{p,q\geq 0} X_{p,q} \times |\Delta^p| \times |\Delta^q| / \sim$$

where $\sim$ is generated by the $(x, t_1, t_2) \sim (y, u_1, u_2)$ whenever $(\partial_{i,j})(x) = y$, $D^i(u_1) = t_1$ and $D^j(u_2) = t_2$. This is equivalent to taking the geometric realisation of the semi-simplicial set first in the p direction, and then the q, or the q followed by the p. This is due to the following homeomorphisms [**21**, 1.9,1.10]

$$\|X_{\bullet,\bullet}\| \cong \|X_{\bullet,q} : [p] \mapsto \|X_\bullet : [q] \mapsto X_{p,q}\|\| \cong \|X_{p,\bullet} : [q] \mapsto \|X_\bullet : [p] \mapsto X_{p,q}\|\|.$$

4.5.13. Semi-simplicial constructions using monoids and submonoids. The following description of the geometric bar construction and related definitions loosely follows Chapter 7 of May's *Classifying spaces and fibrations* [**36**].

Definition 4.5.14. Let M be a monoid and X and Y be spaces with a left and right action of M respectively. Then the *bar construction* denoted $B(Y, M, X)$ is the geometric realisation of the semi-simplicial space $B_\bullet(Y, M, X)$ given by

$$B_n(Y, M, X) = Y \times M^n \times X.$$

Elements in $B_n(Y, M, X)$ are written as $y[g_1, \ldots, g_n]x$ for y in Y, g_i in M for $1 \leq i \leq n$ and x in X. Face maps are given by

$$\partial_i(y[g_1, \ldots, g_n]x) = \begin{cases} yg_1[g_2, \ldots, g_n]x & \text{if } i = 0 \\ y[g_1, \ldots, g_ig_{i+1}, \ldots, g_n]x & \text{if } 1 \leq i \leq n-1 \\ y[g_1, \ldots, g_{n-1}]g_nx & \text{if } i = n. \end{cases}$$

Definition 4.5.15. Consider the bar construction $B(*, M, Y)$ for Y space with an action of the monoid M on the left, and $*$ is a point, on which M acts trivially. This is the geometric realisation of the semi-simplicial set $B_\bullet(*, M, Y)$ given by

$$B_j(*, M, Y) = * \times M^j \times Y.$$

Elements are written as $[m_1, \ldots, m_j]y$ for m_i in M for $1 \leq i \leq j$ and y in Y. Face maps are given by

$$\partial_i([m_1, \ldots, m_j]y) = \begin{cases} [m_2, \ldots, m_j]y & \text{if } i = 0 \\ [m_1, \ldots, m_i m_{i+1}, \ldots, m_j]y & \text{if } 1 \leq i \leq j-1 \\ [m_1, \ldots, m_{j-1}]m_j y & \text{if } i = j. \end{cases}$$

We call the associated bar construction the *homotopy quotient* of Y over M and denote it $B(*, M, Y) = M \mathbin{\backslash\!\!\backslash} Y$. When we have a monoid M acting on a space Y on the right we define the homotopy quotient to be $B(Y, M, *) = Y /\!\!/ M$.

EXAMPLE 4.5.16. Consider the bar construction $B(*, N, M)$ for N a submonoid of M acting on M on the left, by left multiplication, and $*$ a point, on which N acts trivially. Then the homotopy quotient of M over N is $B(*, N, M) = N \mathbin{\backslash\!\!\backslash} M$. This is the geometric realisation of the semi-simplicial set $B_\bullet(*, N, M)$ given by

$$B_j(*, N, M) = * \times N^j \times M.$$

Elements are written as $[n_1, \ldots, n_j]m$ for n_i in N for $1 \leq i \leq j$ and m in M. Face maps are given by

$$\partial_i([n_1, \ldots, n_j]m) = \begin{cases} [n_2, \ldots, n_j]m & \text{if } i = 0 \\ [n_1, \ldots, n_i n_{i+1}, \ldots, n_j]m & \text{if } 1 \leq i \leq j-1 \\ [n_1, \ldots, n_{j-1}]n_j m & \text{if } i = j. \end{cases}$$

We can build a similar homotopy quotient for a submonoid N acting on M on the right by right multiplication. Then the associated homotopy quotient is the geometric realisation $B(M, N, *) = M /\!\!/ N$.

LEMMA 4.5.17. *The homotopy quotient of a group G or monoid M under a point $*$ is a model for the classifying space of the group or monoid, i.e. $BG \simeq G \mathbin{\backslash\!\!\backslash} * \simeq * /\!\!/ G$ and $BM \simeq M \mathbin{\backslash\!\!\backslash} * \simeq * /\!\!/ M$.*

PROOF. Writing down the simplices and face maps for the homotopy quotients $G \mathbin{\backslash\!\!\backslash} *$ and $G /\!\!/ *$ gives exactly the simplices and face maps for the *standard resolution* or *bar resolution* of G, which is a model for BG (see e.g. [**12**]). This holds similarly for the monoid M. In fact in [**36**] this is how the classifying spaces BG and BM are defined. ☐

LEMMA 4.5.18. *For a monoid M, $M \mathbin{\backslash\!\!\backslash} M \simeq *$.*

PROOF. We introduce an augmentation of the semi-simplicial space, as in Definition 4.5.5, by setting $(M \mathbin{\backslash\backslash} M)_{-1} = *$ and the augmentation map $\epsilon_\bullet$ to be the trivial map to the point at each level. By [**21**, Lemma 1.12] the map $\|\epsilon_\bullet\| : M \mathbin{\backslash\backslash} M \to *$ is a homotopy equivalence if there exist maps $h_{p+1} : (M \mathbin{\backslash\backslash} M)_p \to (M \mathbin{\backslash\backslash} M)_{p+1}$ such that:

(1) $\partial_{p+1} h_{p+1} = Id_{(M \mathbin{\backslash\backslash} M)_p}$
(2) $\partial_i h_{p+1} = h_p \partial_i$ for $0 \leq i < p+1$
(3) $\epsilon_0 h_0 = Id_{(M \mathbin{\backslash\backslash} M)_{-1}}$

Letting

$$\begin{array}{rcl} h_{p+1} : (M \mathbin{\backslash\backslash} M)_p & \to & (M \mathbin{\backslash\backslash} M)_{p+1} \\ [m_1, \ldots, m_p]m & \mapsto & [m_1, \ldots, m_p, m]e \end{array}$$

these three hypotheses are easily verified and so $\|\epsilon_\bullet\| : M \mathbin{\backslash\backslash} M \to *$ is a homotopy equivalence. $\square$

LEMMA 4.5.19. *Let N be a monoid and S be a set such that N acts on S on the right. Suppose S can be decomposed as $S \cong X \times Y$ and, under this decomposition, the action of N restricts to a right action on the Y component and trivial action on the X component. Then the homotopy quotient satisfies*

$$S \mathbin{/\!\!/} N \cong (X \times Y) \mathbin{/\!\!/} N \simeq X \times (Y \mathbin{/\!\!/} N)$$

where the homotopy equivalence is given by the geometric realisation of the levelwise map on the bar construction

$$\begin{array}{rcl} B_p((X \times Y), N, *) & \to & X \times B_p(Y, N, *) \\ (x, y)[n_1, \ldots, n_p] & \mapsto & (x, y[n_1, \ldots, n_p]) \end{array}$$

for $x \in X$, $y \in Y$ and $n_i \in N$ for all i.

PROOF. The homotopy quotient $S \mathbin{/\!\!/} N$ is the geometric realisation of the simplicial set $B_\bullet(S, N, *)$ with the set of j-simplices given by

$$B_j(S, N, *) = S \times N^j$$

and face maps given by Definition 4.5.15, the first face map ∂_1 encompassing the right action of N on S. Under the decomposition $S \cong X \times Y$ the set of j-simplices is given by

$$B_j(S, N, *) \cong (X \times Y) \times N^j \cong X \times (Y \times N^j)$$

where the second isomorphism highlights that the action of N on S can be restricted to a right action on Y, since the action is trivial on the X component. Note that the second factor is precisely the set of j-simplices in $B_j(Y, N, *)$, and since the face maps act trivially on the X factor, the face maps in $B_j(S, N, *)$ induce face maps in $B_j(Y, N, *)$ under the decomposition. The proof is concluded by taking the geometric realisation of $B_\bullet(S, N, *)$ and the geometric realisation of $X \times B_\bullet(Y, N, *)$, noting that $\|X \times B_\bullet(Y, N, *)\| \simeq X \times \|B_\bullet(Y, N, *)\|$. $\square$

Given an Artin monoid A^+ and a parabolic submonoid M^+, recall from the previous section that $A^+(M)$ is the set of words in A^+ which do not end in words in M^+ and there is a decomposition as sets (Proposition 4.3.7), $A^+ \cong A^+(M) \times M^+$. This decomposition maps α in A^+ to $(\overline{\alpha}, \beta)$ where $\alpha = \overline{\alpha}\beta$ (as defined in Remark 4.3.2) and the right action of M^+ on A^+ descends to a trivial action on $A^+(M)$ and a right action on M^+.

Proposition 4.5.20. *With notation as above, the map*

$$A^+ /\!\!/ M^+ \to A^+(M)$$

which is defined levelwise on the bar construction $B_\bullet(A^+, M^+, *)$ by

$$\begin{array}{rcl} B_p(A^+, M^+, *) & \to & A^+(M) \\ \alpha[m_1, \ldots, m_p] & \mapsto & \overline{\alpha} \end{array}$$

is a homotopy equivalence.

Proof. From Proposition 4.3.7 $A^+ \cong A^+(M) \times M^+$ and this decomposition respects the right action of M^+ on A^+. Then

$$\begin{array}{rcl} A^+ /\!\!/ M^+ & = & (A^+(M) \times M^+) /\!\!/ M^+ \\ & \simeq & A^+(M) \times (M^+ /\!\!/ M^+) \\ & \simeq & A^+(M) \times * \\ & = & A^+(M) \end{array}$$

where the first homotopy equivalence uses Lemma 4.5.19 and the second homotopy equivalence uses Lemma 4.5.18. The levelwise map given by these two lemmas is precisely the map in the statement. ☐

Proposition 4.5.21. *Let A^+ be a monoid and M^+ be a submonoid. Consider two maps f and $g : A^+ \backslash\!\!\backslash A^+ \to A^+ \backslash\!\!\backslash A^+$ which are equivariant with respect to the action of M^+ on the right of $A^+ \backslash\!\!\backslash A^+$. Since $A^+ \backslash\!\!\backslash A^+ \simeq *$ it follows that f and g are homotopic. We show that there exists an M^+ equivariant homotopy between the two maps.*

Proof. Let the k-cell of $A^+ \backslash\!\!\backslash A^+$ corresponding to geometric realisation of the k-simplex $[p_1, \ldots, p_k]a$ of $B_k(*, A^+, A^+)$ (as described in 4.5.15) be denoted by the tuple $(p_1, \ldots, p_k, a)$, with p_i and a in A^+. There is a right action of A^+ on the k-cells given by $(p_1, \ldots, p_k, a) \cdot \mu = (p_1, \ldots, p_k, a\mu)$. Define the set of *elementary k-cells* to be those with tuple $(p_1, \ldots, p_k, e)$ where e is the identity element in the monoid, and denote this cell $D(p_1, \ldots, p_k)$. Then every k-cell is uniquely determined by an elementary k-cell and an element a in A^+, since $(p_1, \ldots, p_k, a) = D(p_1, \ldots, p_k) \cdot a$. Denote the set of k-cells in $A^+ \backslash\!\!\backslash A^+$ as $(A^+ \backslash\!\!\backslash A^+)_k$. The isomorphism of 4.3.7, gives that $A^+ = A^+(M) \times M^+$, let $a = \bar{a}m$ under this decomposition.

Then we get the following description for k-cells

$$\begin{array}{lll} (A^+ \doublebackslash A^+)_k \cong \coprod_{(p_1,\ldots,p_k)} D(p_1,\ldots,p_k) \times A^+ & \cong & \coprod_{(p_1,\ldots,p_k)} D(p_1,\ldots,p_k) \times (A^+(M) \times M^+). \\ (p_1,\ldots,p_k,a) \mapsto (D(p_1,\ldots,p_k),a) & & \mapsto (D(p_1,\ldots,p_k),(\bar{a},m)) \end{array}$$

We build the equivariant homotopy first for 0-cells in $A^+ \doublebackslash A^+$ and then inductively, showing if we have built an equivariant homotopy on the $(k-1)$-skeleton we can extend it to the k-cells. Let f_k be the restriction of the map f to the k-cells of $A^+ \doublebackslash A^+$ and similarly for g_k. Then we aim to define an equivariant homotopy between f_0 and g_0. $(A^+ \doublebackslash A^+)_0 \cong (A^+(M) \times M^+)$ under the above decomposition. Consider $f_0(\overline{\alpha})$ and $g_0(\overline{\alpha})$ in $A^+ \doublebackslash A^+$ for some $\overline{\alpha}$ in $A^+(M)$. Then since $A^+ \doublebackslash A^+ \simeq *$ by 4.5.18 it follows that there exists a path between $f_0(\overline{\alpha})$ and $g_0(\overline{\alpha})$: call this $h_0(\overline{\alpha},t)$ for $t \in [0,1]$. Extend this homotopy to all 0-cells by letting $h_0(\overline{\alpha}m,t) = h_0(\overline{\alpha},t)\cdot m$ for all m in M^+. Then $h_0(\overline{\alpha}m,0) = h_0(\overline{\alpha},0)\cdot m = f_0(\overline{\alpha})\cdot m = f_0(\overline{\alpha}m)$ and similarly $h_0(\overline{\alpha}m,1) = h_0(\overline{\alpha},1)\cdot m = g_0(\overline{\alpha})\cdot m = g_0(\overline{\alpha}m)$, since f_0 and g_0 are M^+ equivariant. The homotopy $h_0(x,t)$ is M^+ equivariant, since $h_0(x,t)\cdot\mu = h_0(\bar{x}m,t)\cdot\mu = h_0(\bar{x},t)\cdot m\mu = h_0(\bar{x}m\mu,t) = h_0(x\mu,t)$ when the decomposition of x is given by $x = \bar{x}m$ for some $\bar{x}$ in $A^+(M)$ and m in M^+.

We now assume that we have built the equivariant homotopy $h_{k-1}(x,t)$ on the $(k-1)$-skeleton and show that we extend it to the k-cells. The homotopy $h_{k-1}(x,t)$ satisfies $h_{k-1}(x,0) = f_{k-1}(x)$ and $h_{k-1}(x,1) = g_{k-1}(x)$. Consider the k-cell $D(p_1,\ldots,p_k)\cdot\overline{\alpha}$ for some $\overline{\alpha}$ in $A^+(M)$. Then its boundary consists of $(k-1)$-cells and it follows that h_{k-1} defines a homotopy

$$(\partial(D(p_1,\ldots,p_k))\cdot\overline{\alpha})\times I \to A^+ \doublebackslash A^+$$

and the maps f_k and g_k also define maps

$$\begin{aligned} f_k : ((D(p_1,\ldots,p_k))\cdot\overline{\alpha})\times\{0\} &\to A^+ \doublebackslash A^+ \\ g_k : ((D(p_1,\ldots,p_k))\cdot\overline{\alpha})\times\{1\} &\to A^+ \doublebackslash A^+. \end{aligned}$$

The union of these three maps gives a map from the boundary of $(D(p_1,\ldots,p_k)\cdot\overline{\alpha})\times I$ to $A^+ \doublebackslash A^+$, but this boundary is a k-sphere and so, since $A^+ \doublebackslash A^+$ is contractible the k-sphere bounds a $(k+1)$-disk. We can compatibly extend the map over this disk to create the required homotopy

$$h_k : (D(p_1,\ldots,p_k)\cdot\overline{\alpha})\times I \to A^+ \doublebackslash A^+$$

which agrees on the boundary with the three maps above. Now define h_k on any k-cell $D(p_1,\ldots,p_k)\cdot\overline{\alpha}m$ by letting $h_k(x\cdot m,t) = h_k(x,t)\cdot m$ for x in $D(p_1,\ldots,p_k)\cdot\overline{\alpha}$. Then h_k is M^+ equivariant by construction, and satisfies $h_k(x,0) = f_k$ and $h_k(x,1) = g_k$ by construction and the fact that both f_k and g_k are M^+ equivariant. $\square$

Definition 4.5.22. Given a monoid M and two submonoids N_1 and N_2 we can define the *double homotopy quotient* $N_1 \doublebackslash M /\!\!/ N_2$ to be the realisation of the bi-semi-simplicial set given by taking two simplicial directions relating to bar constructions $B_\bullet(*, N_1, M)$ and $B_\bullet(M, N_2, *)$. The p, q level of the associated bi-semi-simplicial set $X_{\bullet\bullet}$ has simplices

$$X_{p,q} = N_1^p \times M \times N_2^q$$

and face maps inherited from $B_\bullet(*, N_1, M)$ in the p direction ($\partial_{p,\bullet}$) and $B_\bullet(M, N_2, *)$ in the q direction ($\partial_{\bullet,q}$). An element in the p, q level is given by $[n_1, \ldots n_p]m[n'_1, \ldots, n'_q]$ with n_i in N_1 and n'_j in N_2 for $1 \leq i \leq p$ and $1 \leq j \leq q$. We note that the face maps on the left and right commute, since the only maps which act on the same coordinates are $\partial_{p,\bullet}$ in the p direction and $\partial_{\bullet,1}$ in the q direction and these commute as follows:

$$\begin{aligned}
\partial_{p,\bullet}(\partial_{\bullet,1}([n_1, \ldots n_p]m[n'_1, \ldots, n'_q])) &= \partial_{p,\bullet}([n_1, \ldots n_p]mn'_1[n'_2, \ldots, n'_q]) \\
&= [n_1, \ldots, n_{p-1}]n_p m n'_1[n'_2, \ldots, n'_q] \\
&= \partial_{\bullet,1}(\partial_{p,\bullet}([n_1, \ldots n_p]m[n'_1, \ldots, n'_q])).
\end{aligned}$$

CHAPTER 5

Background: Homological stability

5.1. Definition and examples

DEFINITION 5.1.1. A family of groups or monoids

$$G_1 \to G_2 \to \cdots \to G_n \to \cdots$$

is said to satisfy *homological stability* if the induced maps on homology

$$H_i(G_n) \to H_i(G_{n+1})$$

are isomorphisms for n sufficiently large compared to i.

Homological stability has been proved in a variety of cases e.g. for the symmetric groups, braid groups, general linear groups and mapping class groups of surfaces. We will now focus on some of these examples in detail.

EXAMPLE 5.1.2. The sequence of symmetric groups S_n satisfies homological stability, as first proved by Nakaoka [**39**]. There is a sequence of groups and inclusions:

$$S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow \cdots$$

where the inclusion $S_n \hookrightarrow S_{n+1}$ is given by extending a permutation of n elements to a permutation of $(n+1)$ elements by fixing the last element. Then homological stability:

$$H_i(S_n) \xrightarrow{\cong} H_i(S_{n+1})$$

holds in the range $2i \leq n$.

EXAMPLE 5.1.3. For the braid group on n strands, $\mathcal{B}_n$, we have a sequence of groups:

$$\mathcal{B}_1 \hookrightarrow \mathcal{B}_2 \hookrightarrow \ldots \hookrightarrow \mathcal{B}_n \hookrightarrow \ldots$$

where the inclusion $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ is given by adding a strand in the $(n+1)$st position which does not entangle with the first n strands. The sequence of braid groups $\mathcal{B}_n$ satisfies homological stability, as first proved by Arnol'd and published by Brieskorn [**10**]. For this sequence we have:

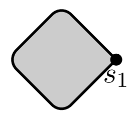
$$H_i(\mathcal{B}_n) \xrightarrow{\cong} H_i(\mathcal{B}_{n+1})$$

holds in the range $2i \leq n$.

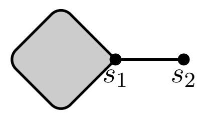
5.2. Homological stability for Coxeter groups

This section follows work of Hepworth [**31**], which inspired the project which comprises the next chapter of this thesis.

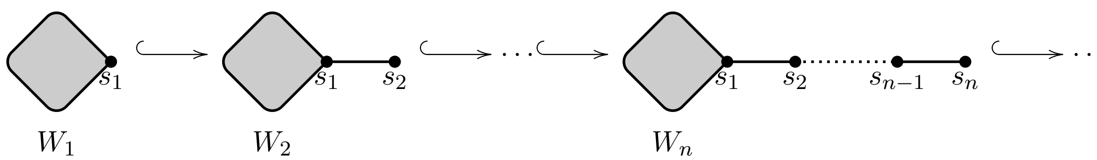
Hepworth proves homological stability for families of Coxeter groups for which the sequence of groups and inclusions is constructed as follows. The first group in the sequence, W_1 is given by any Coxeter diagram $\mathcal{D}_{W_1}$, and a vertex of this diagram, i.e. an s_1 in S, is chosen:



the next group in the sequence, W_2 is built by adding a generator s_2 such that $m(s_1, s_2) = 3$ and s_2 commutes with all other generators of W_1, i.e. the Coxeter diagram has the form



Since the diagram $\mathcal{D}_{W_1}$ is a subdiagram of $\mathcal{D}_{W_2}$ it follows that W_1 is a subgroup of W_2. We continue in this sense, at each stage progressing from W_i to W_{i+1} by adding a generator s_{i+1} satisfying $m(s_i, s_{i+1}) = 3$ and s_{i+1} commutes with all other generators of W_i. At each stage the Coxeter diagram $\mathcal{D}_{W_i}$ is a full subdiagram of $\mathcal{D}_{W_{i+1}}$ and hence W_i is a subgroup W_{i+1}, by Proposition 1.2.5. Therefore the sequence $\{W_n\}$ has the following form:



We note here that the Coxeter diagram $\mathcal{D}_{W_n}$ has the diagram A_n as a subdiagram, and so the finite Coxeter group $W(A_n)$ is a subgroup of W_n. Recall that $W(A_n)$ corresponds to the symmetric group S_{n+1}. Therefore each group in the sequence has a symmetric group as a subgroup, and the 'dimension', or the number of generators, in the symmetric group increases as one moves up the sequence. Hepworth's result is as follows:

THEOREM 5.2.1 (Hepworth [**31**, Main Theorem]). *The above sequence of groups and inclusions*

$$W_1 \hookrightarrow W_2 \hookrightarrow \cdots \hookrightarrow W_n \hookrightarrow \cdots$$

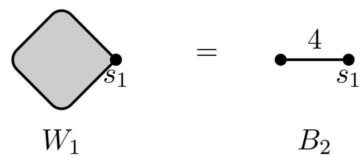
satisfies homological stability, that is the induced map on homology

$$H_*(BW_{n-1}) \to H_*(BW_n)$$

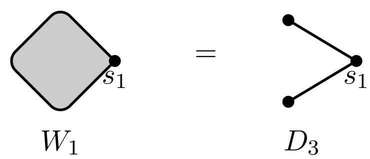
is an isomorphism for $2* \leq n$. Here homology is taken with arbitrary constant coefficients.

Three of the families of finite Coxeter groups from Theorem 1.1.12 satisfy that their diagrams are of the form of Hepworth's construction. These are:

- $W(A_n)$, or the symmetric group S_{n+1}, which relates to the sequence $\{W_n\}$ by setting $W_i = W(A_i)$. In this case the starting diagram $\mathcal{D}_{W_1}$ is given by the single vertex s_1, or the diagram A_1. This gives homological stability for the symmetric groups, as in Example 5.1.2.
- $W(B_n)$, or the signed symmetric groups $\mathbb{Z}_2 \wr S_n$ relates to the sequence $\{W_n\}$ by setting $W_i = W(B_{i+1})$. In this case the starting diagram $\mathcal{D}_{W_1}$ is given by the diagram B_2, as follows:

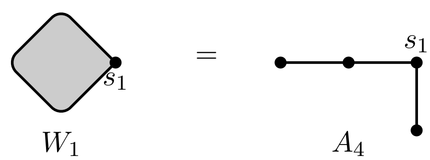


  Homological stability was proved for wreath products by Hatcher and Wahl in [**30**, Proposition 1.6].
- $W(D_n)$, or the index two subgroup of $W(B_n)$, which relates to the sequence $\{W_n\}$ by setting $W_i = W(D_{i+2})$. In this case the starting diagram $\mathcal{D}_{W_1}$ is given by the diagram D_3, as follows:



  This was a previously unknown homological stability result.

REMARK 5.2.2. Since the Coxeter diagram $\mathcal{D}_{W_1}$ can be any diagram with a finite number of vertices, Hepworth's result also proves homological stability for sequences of infinite Coxeter groups, and for cases when the sequence is neither comprised fully of finite nor of infinite groups. For example, in the case that the starting diagram is as follows:



Then the first five groups in the sequence are finite and the sequence takes the form

$$W(A_4) \hookrightarrow W(D_5) \hookrightarrow W(E_6) \hookrightarrow W(E_7) \hookrightarrow W(E_8) \hookrightarrow \cdots$$

however after the fifth group, the groups in the sequence become infinite Coxeter groups.

5.3. Homological stability for Artin groups: literature review

Inspired by the work of Hepworth described in the previous section, we aim to prove a homological stability result for the sequence of Artin groups $\{A_{W_n}\}$ corresponding to the sequence of Coxeter groups $\{W_n\}$ of Hepworth's paper. There are a few known cases of stability for sequences of this form, reinforcing the hypothesis that a general statement such as Hepworth's will hold. All of the following examples were proved by Arnol'd, by computing the full (co)homology of the groups in question, using the associated hyperplane complement. The results and proofs are in the paper *Sur les groupes des tresses* by Brieskorn [**10**].

- Homological stability holds for the braid groups, by Example 5.1.3. This is the sequence of Artin groups $\{A_{W_n}\}$ for W_n the symmetric group $W(A_n) = S_{n+1}$.
- Homological stability holds for the sequence of finite type Artin groups $\{A_{W_n}\}$ relating to W_n being the Coxeter group $W(B_{n+1})$.
- Homological stability holds for the sequence of finite type Artin groups $\{A_{W_n}\}$ relating to W_n being the Coxeter group $W(D_{n+2})$.

These examples are exactly the sequences of finite type Artin groups relating to the three sequences of finite Coxeter groups known to fit into Hepworth's result. However Hepworth's result is much more general and this is what we aim to prove in the case of Artin groups.

In *Second Mod 2 Homology of Artin Groups* by Akita and Liu [**1**], homological stability in degree two with $\mathbb{Z}_2$ coefficients was proved, for the sequence of Artin groups $\{A_{W_n}\}$ relating to Hepworth's sequence $\{W_n\}$. They proved this by showing the mod 2 homology in degree two of any finite rank Artin group was isomorphic to the mod 2 homology of the corresponding Coxeter group. Therefore Howlett's Theorem or Theorem A gives $H_2(A; \mathbb{Z}_2)$, and they observe that for the sequence of diagrams relating to Hepworth's, this formula stabilises.

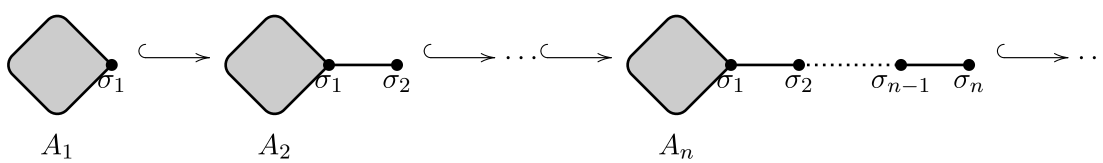
CHAPTER 6

Results: Homological stability for Artin monoids

In this chapter we prove a homological stability result for families of Artin monoids corresponding to Hepworth's families of Coxeter groups. The key step in the proof of the theorem is to show that a certain family of semi-simplicial spaces on which the monoids act is highly connected. To define this family of spaces and prove the related connectivity requires the theory of the previous chapter.

6.1. Discussion of results

This chapter concerns the homological stability behaviour of families of Artin groups. In particular we consider sequences of Artin groups which have the braid group as a subgroup. The sequence of groups and inclusions relates to the sequence of Coxeter groups $\{W_n\}_{n\geq 1}$ introduced by Hepworth [**31**], and described in Section 5.2. We let the Artin group A_{W_n} corresponding to the Coxeter group W_n be denoted A_n, for ease of notation the sequence of corresponding diagrams is



where the grey box indicates a diagram of arbitrary shape, meaning that the sequence begins with an arbitrary Artin group with finite generating set. As in the Coxeter group setting, this gives a sequence of groups and inclusions

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots .$$

The finite type examples of this sequence were discussed in Section 5.3 and are known to satisfy homological stability. The results in this section relate to the more general setting, where A_1 can correspond to any Coxeter diagram, but are stated and proved for the corresponding Artin monoids. The results are then related to Artin groups via the $K(\pi, 1)$ conjecture, discussed in Section 3.2.

Recall that the Artin monoid corresponding to A_n is denoted A_n^+. The inclusion map between the monoids is denoted s and called the *stabilisation map*. This gives the following sequence of monoids, studied in this chapter.

$$A_1^+ \overset{s}{\hookrightarrow} A_2^+ \overset{s}{\hookrightarrow} \cdots \overset{s}{\hookrightarrow} A_n^+ \overset{s}{\hookrightarrow} \cdots$$

THEOREM 6.1.1. *The sequence of Artin monoids*

$$A_1^+ \hookrightarrow A_2^+ \hookrightarrow \cdots \hookrightarrow A_n^+ \hookrightarrow \cdots$$

satisfies homological stability. That is, the induced map on homology

$$H_*(BA_{n-1}^+) \xrightarrow{s_*} H_*(BA_n^+)$$

is an isomorphism when $* < \frac{n}{2}$ *and a surjection when* $* = \frac{n}{2}$*. Here homology is taken with arbitrary constant coefficients.*

Recall from Theorem 4.4.1 that the $K(\pi, 1)$ conjecture holds precisely when the classifying spaces of the Artin group and monoid are homotopy equivalent. Hence, if the conjecture holds, Theorem 6.1.1 implies homological stability even for the groups.

COROLLARY 6.1.2. *When the* $K(\pi, 1)$ *conjecture holds for all* A_n*, the sequence of Artin groups*

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots$$

satisfies homological stability. That is, the induced map on homology

$$H_*(BA_{n-1}) \to H_*(BA_n)$$

is an isomorphism when $* < \frac{n}{2}$ *and a surjection when* $* = \frac{n}{2}$*. Here homology is taken with arbitrary constant coefficients.*

PROOF. We have by Theorem 4.4.1 that the $K(\pi, 1)$ conjecture holds if and only if $BA^+ \simeq BA$ via the natural map. Applying this homotopy equivalence to Theorem 6.1.1 yields the Corollary. $\square$

This in turn reproves the homological stability results in Section 5.3.

COROLLARY 6.1.3. *Homological stability holds for the sequences of Artin groups* $\{A_n\}_{n\geq 1}$ *relating to the sequences of finite Coxeter groups* $W(A_n)$, $W(B_{n+1})$ *and* $W(D_{n+2})$.

PROOF. These three sequences consist of only finite type Artin groups, which satisfy the $K(\pi, 1)$ conjecture by Deligne's Theorem (Theorem 3.2.5). Hence by the previous Corollary, the sequences of Artin groups satisfy homological stability. $\square$

6.2. Outline of proof

The key step in the proof of Theorem 6.1.1 is to show that a certain family of semi-simplicial spaces on which the monoids A_n^+ act is highly connected. In this proof we build a semi-simplicial space $\mathcal{A}_\bullet^n$ for each monoid in the sequence A_n^+ such that:

(1) $\mathcal{A}_\bullet^n$ is built out of spaces $\mathcal{A}_p^n$ for $p \geq 0$
(2) there exist homotopy equivalences $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ for $p \geq 0$
(3) there is a map from the geometric realisation of $\mathcal{A}_\bullet^n$ to the classifying space BA_n^+, which we call $\|\phi_\bullet\|$

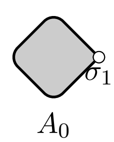
$$\|\mathcal{A}_\bullet^n\| \xrightarrow{\|\phi_\bullet\|} BA_n^+$$

(4) $\|\phi_\bullet\|$ is highly connected, i.e. it is an isomorphism on a large range of homotopy groups.

We will refer to these four points as 1, 2, 3 and 4 throughout this chapter, and address each point in turn. In this chapter the sections are arranged as follows. Section 6.3 applies the theory of Section 4.3 in the case of the sequence of monoids we are working with, and introduces notation used throughout the chapter. Section 6.4 introduces the semi-simplicial space $\mathcal{A}_\bullet^n$ for each monoid in the sequence A_n^+ and addresses Points 1, 2 and 3. Point 4 is then the topic of Section 6.5, in which the general method of proof for the high connectivity argument is introduced before the proof is split into cases which are then proved individually. Finally the homological stability result follows in Section 6.6.

6.3. Preliminaries concerning the sequence of Artin monoids

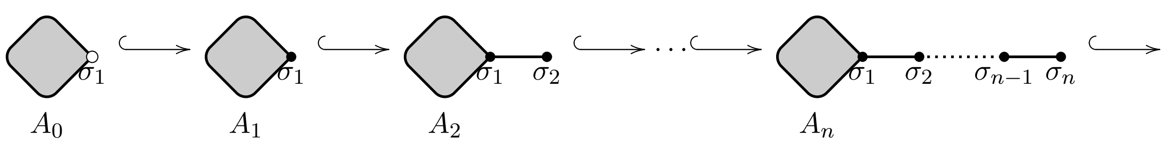
DEFINITION 6.3.1. Let A_0 be the Artin group corresponding to the Coxeter diagram $\mathcal{D}_{W_1}$, but with the vertex s_1 and all edges which have vertex s_1 at one end removed. We depict the diagram as follows



Then $A_0 \hookrightarrow A_1$ and we consider the sequence of Artin monoids

$$A_0^+ \hookrightarrow A_1^+ \hookrightarrow A_2^+ \hookrightarrow \cdots \hookrightarrow A_n^+ \hookrightarrow \cdots \tag{5}$$

given by the diagrams



Here we note that for all p, every generator and hence every word in the monoid A_p^+ commutes with σ_j for $j \geq p+1$.

We now apply the theory developed in Section 4.3 to the specific case of A_n^+ a monoid in the sequence from Equation (5) and a submonoid of A_n^+, given by a previous monoid in the sequence A_p^+ where $p < n$. We adopt the following notation:

- Let $EndMon_p(\alpha) = EndMon_{A_p}(\alpha)$ and $EndGen_p(\alpha) = EndGen_{A_p}(\alpha)$ for α in A_n^+, as defined in Definition 4.2.9. Then
$$\begin{aligned} EndGen_p(\alpha) &= \{\sigma_s \mid s \in S_{A_p^+}, \sigma_s \preceq_R \alpha\} \\ EndMon_p(\alpha) &= \{\beta \in A_p^+ \mid \beta \preceq_R \alpha\}. \end{aligned}$$
- Let $A^+(n;p)$ be the set $A^+(M)$ for $A^+ = A_n^+$ and $M = A_p^+$. This set is defined in Definition 4.3.3, and is the set of words in A_n^+ that do not end in a word from A_p^+.
- Let the equivalence class of α in A_n^+ under the relation $\approx$ with respect to the submonoid A_p^+ (defined in Definition 4.3.5) be denoted $[\alpha]_p$ as opposed to $[\alpha]_{A_p}$. Then $[\alpha]_p$ is the equivalence class of α under $\approx$, which is the equivalence relation generated by the transitive closure of the relation $\sim$ on A_n^+ given by
$$\alpha_1 \sim \alpha_2 \iff \alpha_1\beta_1 = \alpha_2\beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } A_p^+.$$

Then we have from Lemma 4.3.6 that the equivalence classes under $\approx$ with respect to the submonoid A_p^+ are in one to one correspondence with the set $A^+(n;p)$. Recall from Remark 4.3.2 that if β is the least common multiple of $EndMon_p(\alpha)$ then we define $\overline{\alpha}$ in A_n^+ to be the word such that $\alpha = \overline{\alpha}\beta$. Then $A^+(n;p)$ is the set of all such $\overline{\alpha}$ and for all α_1 and α_2 in A_n^+:
$$[\alpha_1]_p = [\alpha_2]_p \iff \overline{\alpha_1} = \overline{\alpha_2}.$$
We also have from Proposition 4.3.7 the set decomposition
$$A_n^+ \cong A^+(n;p) \times A_p^+ \text{ for all } p < n.$$

6.4. The semi-simplicial space $\mathcal{A}^n_\bullet$

We now build the semi-simplicial space $\mathcal{A}^n_\bullet$ as required in Section 6.2 Point 1: $\mathcal{A}^n_\bullet$ is built out of spaces $\mathcal{A}^n_p$ for $p \geq 0$.

Definition 6.4.1. We define the semi-simplicial set $\mathcal{C}^n_\bullet$ by setting levels $\mathcal{C}^n_p$ for $0 \leq p \leq (n-1)$ to be the equivalence classes $A_n^+/\approx$ where the equivalence relation is taken with respect to A^+_{n-p-1}, i.e. $\approx$ is the transitive closure of the relation $\sim$ on A_n^+ given by
$$\alpha_1 \sim \alpha_2 \iff \alpha_1\beta_1 = \alpha_2\beta_2 \text{ for some } \beta_1 \text{ and } \beta_2 \text{ in } A^+_{n-p-1}.$$
Face maps are given by
$$\begin{aligned} \partial^p_k : \mathcal{C}^n_p &\to \mathcal{C}^n_{p-1} \\ \partial^p_k : [\alpha]_{n-p-1} &\mapsto [\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p}. \end{aligned}$$

These face maps are well defined, if $[\alpha]_{n-p-1} = [\beta]_{n-p-1}$ then $\overline{\alpha} = \bar{\beta}$ where the bar is taken with respect to A^+_{n-p-1}. Set $\overline{\alpha} = \gamma$ (recall the definition of $\overline{\alpha}$ from Remark 4.3.2). It follows there exist some a and b in A^+_{n-p-1} such that $\alpha = \gamma a$ and $\beta = \gamma b$. Then since a and b only contain letters in A^+_{n-p-1} and all of these letters commute with $(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})$ it follows that a and b letterwise commute with the face map. Taking the equivalence classes with respect to A^+_{n-p} therefore gives

$$\begin{aligned} & [\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \\ = \; & [(\gamma a)(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \\ = \; & [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})a]_{n-p} \\ = \; & [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \end{aligned}$$

and similarly

$$\begin{aligned} & [\beta(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \\ = \; & [(\gamma b)(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \\ = \; & [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})b]_{n-p} \\ = \; & [\gamma(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})]_{n-p} \end{aligned}$$

therefore α and β map to the same equivalence class under the face map, and so the face maps are well defined. The motivation for this choice of face maps follows Hepworth, as discussed in [**31**, Example 35].

LEMMA 6.4.2. *The face maps ∂^p_k on $\mathcal{C}^n_\bullet$ defined in Definition 6.4.1 satisfy the simplicial identities, that is, for $0 \le i < j \le p$:*

$$\partial^{p-1}_i \partial^p_j = \partial^{p-1}_{j-1}\partial^p_i$$

PROOF. For ease of notation in the proof, we denote $(n-p)$ as r. Then the left hand side acts as follows

$$\begin{array}{ccccc} \mathcal{C}^n_p & \xrightarrow{\partial^p_j} & \mathcal{C}^n_{p-1} & \xrightarrow{\partial^{p-1}_i} & \mathcal{C}^n_{p-2} \\ [\alpha]_{r-1} & \xmapsto{\partial^p_j} & [\alpha(\sigma_{r+j}\dots\sigma_{r+1})]_r & \xmapsto{\partial^{p-1}_i} & [\alpha(\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i+1}\dots\sigma_{r+2})]_{r+1}. \end{array}$$

In comparison the right hand side acts as follows

$$\begin{array}{ccccc} \mathcal{C}^n_p & \xrightarrow{\partial^p_i} & \mathcal{C}^n_{p-1} & \xrightarrow{\partial^{p-1}_{j-1}} & \mathcal{C}^n_{p-2} \\ [\alpha]_{r-1} & \xmapsto{\partial^p_i} & [\alpha(\sigma_{r+i}\dots\sigma_{r+1})]_r & \xmapsto{\partial^{p-1}_{j-1}} & [\alpha(\sigma_{r+i}\dots\sigma_{r+1})(\sigma_{r+j}\dots\sigma_{r+2})]_{r+1} \end{array}$$

Claim: Let $x = (\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i+1}\dots\sigma_{r+2})$ and $y = (\sigma_{r+i}\dots\sigma_{r+1})(\sigma_{r+j}\dots\sigma_{r+2})$. Then $x = y\sigma_{r+1}$.

If we prove the claim then it follows that the left hand side is equal to the right hand side since we are taking the equivalence relation with respect to the submonoid A^+_{r+1}. It therefore remains to prove the claim, which is pure manipulation of the words in the monoid, using the braiding relations.

$$\begin{aligned}
x &= (\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i+1}\dots\sigma_{r+2})\\
&= (\sigma_{r+j}\dots\sigma_{r+i})(\sigma_{r+i-1}\dots\sigma_{r+1})(\sigma_{r+i+1}\sigma_{r+i}\dots\sigma_{r+2})\\
&= (\sigma_{r+j}\dots\sigma_{r+i+1}\sigma_{r+i})\sigma_{r+i+1}(\sigma_{r+i-1}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= (\sigma_{r+j}\dots\sigma_{r+i+2})(\sigma_{r+i+1}\sigma_{r+i}\sigma_{r+i+1})(\sigma_{r+i-1}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= (\sigma_{r+j}\dots\sigma_{r+i+2})(\sigma_{r+i}\sigma_{r+i+1}\sigma_{r+i})(\sigma_{r+i-1}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= \sigma_{r+i}(\sigma_{r+j}\dots\sigma_{r+i+2}\sigma_{r+i+1}\sigma_{r+i})(\sigma_{r+i-1}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= \sigma_{r+i}(\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= \sigma_{r+i}(\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i}\dots\sigma_{r+2})\\
&= \sigma_{r+i}\sigma_{r+i-1}(\sigma_{r+j}\dots\sigma_{r+1})(\sigma_{r+i-1}\dots\sigma_{r+2})\\
&= \cdots\\
&= (\sigma_{r+i}\sigma_{r+i-1}\dots\sigma_{r+2})(\sigma_{r+j}\dots\sigma_{r+2}\sigma_{r+1})(\sigma_{r+2})\\
&= (\sigma_{r+i}\sigma_{r+i-1}\dots\sigma_{r+2})(\sigma_{r+j}\dots\sigma_{r+3})(\sigma_{r+2}\sigma_{r+1}\sigma_{r+2})\\
&= (\sigma_{r+i}\sigma_{r+i-1}\dots\sigma_{r+2})(\sigma_{r+j}\dots\sigma_{r+3})(\sigma_{r+1}\sigma_{r+2}\sigma_{r+1})\\
&= (\sigma_{r+i}\dots\sigma_{r+2}\sigma_{r+1})(\sigma_{r+j}\dots\sigma_{r+3})(\sigma_{r+2}\sigma_{r+1})\\
&= (\sigma_{r+i}\dots\sigma_{r+2}\sigma_{r+1})(\sigma_{r+j}\dots\sigma_{r+3}\sigma_{r+2}\sigma_{r+1})\\
&= (\sigma_{r+i}\dots\sigma_{r+1})(\sigma_{r+j}\dots\sigma_{r+2})\sigma_{r+1}\\
&= y\sigma_{r+1}.
\end{aligned}$$

□

LEMMA 6.4.3. *The pth level of $\mathcal{C}^n_\bullet$ satisfies*

$$A^+_n /\!\!/ A^+_{n-p-1} \simeq A^+(n;n-p-1) = \mathcal{C}^n_p$$

where $A^+(n;n-p-1)$ is a defined at the beginning of this section. The homotopy equivalence is given by the map defined levelwise on the bar construction by

$$\begin{aligned}
B_p(A^+_n, A^+_{n-p-1}, *) &\to A^+(n;n-p-1)\\
\alpha[m_1,\dots,m_p] &\mapsto \overline{\alpha}
\end{aligned}$$

where $\alpha \in A^+_n$, $m_i \in A^+_{n-p-1}$ for all i and $\alpha = \overline{\alpha}\beta$ for $\overline{\alpha} \in A^+(n;n-p-1)$ and $\beta \in A^+_{n-p-1}$.

PROOF. This is a direct application of Proposition 4.5.20 and the decomposition $A^+_n \cong A^+(n;n-p-1) \times A^+_{n-p-1}$. □

Definition 6.4.4. Let the semi-simplicial space $\mathcal{A}^n_\bullet$ be the semi-simplicial space with pth level the homotopy quotient $\mathcal{A}^n_p = A^+_n \sslash \mathcal{C}^n_p$, where the action of A^+_n on $A^+(n; n-p-1)$ is given by

$$a \cdot [\alpha]_{n-p-1} = [a\alpha]_{n-p-1} \text{ for } a, \alpha \in A^+_n.$$

Then $\mathcal{A}^n_\bullet$ is given by:

$$\begin{array}{ccc}
A^+_n \sslash \mathcal{C}^n_{n-1} & & \mathcal{A}^n_{n-1} \\
\downarrow\downarrow \cdots \downarrow\downarrow & & \\
A^+_n \sslash \mathcal{C}^n_{n-2} & & \mathcal{A}^n_{n-2} \\
\downarrow\downarrow \cdots \downarrow\downarrow & & \\
A^+_n \sslash \mathcal{C}^n_{n-3} & & \mathcal{A}^n_{n-3} \\
\downarrow\downarrow \cdots \downarrow\downarrow & & \\
\vdots & & \vdots \\
\downarrow\downarrow\downarrow & & \\
A^+_n \sslash \mathcal{C}^n_{1} & & \mathcal{A}^n_{1} \\
\downarrow\downarrow & & \\
A^+_n \sslash \mathcal{C}^n_{0} & & \mathcal{A}^n_{0}
\end{array}$$

where face maps are denoted by ∂^p_k for $0 \leq k \leq p$

$$\begin{aligned}
\partial^p_k : \mathcal{A}^n_p &\to \mathcal{A}^n_{p-1} \\
\partial^p_k : A^+_n \sslash \mathcal{C}^n_p &\to A^+_n \sslash \mathcal{C}^n_{p-1}
\end{aligned}$$

and ∂^p_k acts as the face map ∂^p_k from 6.4.1 on the $\mathcal{C}^n_p$ factor of each simplex in the homotopy quotient, and as the identity on the other factors. The set of j-simplices in $A^+_n \sslash \mathcal{C}^n_p$ is given by $(A^+_n)^j \times \mathcal{C}^n_p$ and an element in this set is given by $[a_1, \ldots, a_j][\alpha]_{n-p-1}$ where the a_i and α are in A^+_n. Then the map ∂^p_k acts on this simplex as

$$\partial^p_k([a_1, \ldots, a_j][\alpha]_{n-p-1}) \mapsto [a_1, \ldots, a_j][\alpha(\sigma_{n-p+k}\sigma_{n-p+k-1} \ldots, \sigma_{n-p+1})]_{n-p}$$

and since the multiplication by $(\sigma_{n-p+k}\sigma_{n-p+k-1} \ldots \sigma_{n-p+1})$ is on the right it follows that ∂^p_k commutes with all face maps of the bar construction $B_\bullet(*, A^+_n, \mathcal{C}^n_p)$ for each k. Therefore the definition of ∂^p_k on the simplicial level gives a map on the homotopy quotient $A^+_n \sslash \mathcal{C}^n_p$.

Lemma 6.4.5. *The face maps ∂^p_k on $\mathcal{A}^n_\bullet$ defined in Definition 6.4.4 satisfy the simplicial identities, that is for $0 \leq i < j \leq p$:*

$$\partial^{p-1}_i \partial^p_j = \partial^{p-1}_{j-1} \partial^p_i.$$

Proof. This proof follows directly from the fact that the simplicial identities are satisfied for $\mathcal{C}^n_\bullet$ (Lemma 6.4.2), since the face maps for $\mathcal{A}^n_\bullet$ are defined via the maps for $\mathcal{C}^n_\bullet$. $\square$

To address Point 2: there exist homotopy equivalences $\mathcal{A}^n_p \simeq BA^+_{n-p-1}$ for $p \geq 0$, in Section 6.2, we prove the following lemma.

LEMMA 6.4.6. *The pth level of the space $\mathcal{A}^n_\bullet$ satisfies*

$$\mathcal{A}^n_p \simeq A^+_n \mathbin{\backslash\!\backslash} A^+_n \mathbin{/\!\!/} A^+_{n-p-1} \simeq BA^+_{n-p-1}$$

where the homotopy equivalence from the central term to the left is given by the realisation of the levelwise map on (j,k)-simplices:

$$\begin{array}{rcl} (A^+_n \mathbin{\backslash\!\backslash} A^+_n \mathbin{/\!\!/} A^+_{n-p-1})_{(j,k)} & \to & (A^+_n \mathbin{\backslash\!\backslash} \mathcal{C}^n_p)_j \\ [a_1,\dots,a_j]\alpha[a'_1,\dots,a'_k] & \mapsto & [a_1,\dots,a_j]\overline{\alpha} \end{array}$$

and the second homotopy equivalence is given by the levelwise projection of the (j,k)-simplices of the double homotopy quotient map to the k-simplices of the single homotopy quotient:

$$\begin{array}{rcl} (A^+_n \mathbin{\backslash\!\backslash} A^+_n \mathbin{/\!\!/} A^+_{n-p-1})_{(j,k)} & \to & (* \mathbin{/\!\!/} A^+_{n-p-1})_k \\ [a_1,\dots,a_j]\alpha[a'_1,\dots,a'_k] & \mapsto & *[a'_1,\dots,a'_k] \end{array}$$

where α and a_i are in A^+_n, a'_i is in A^+_{n-p-1}, and $\alpha = \overline{\alpha}\beta$ for $\overline{\alpha}$ in $A^+(n;n-p-1)$ and β in A^+_{n-p-1}.

PROOF. From Lemma 6.4.3 $\mathcal{C}^n_p = A^+(n;n-p-1) \simeq A^+_n \mathbin{/\!\!/} A^+_{n-p-1}$, and this induces

$$\mathcal{A}^n_p = A^+_n \mathbin{\backslash\!\backslash} \mathcal{C}^n_p \simeq A^+_n \mathbin{\backslash\!\backslash} A^+_n \mathbin{/\!\!/} A^+_{n-p-1}$$

with the homotopy equivalence given by the required map. We then have the following

$$\mathcal{A}^n_p \simeq A^+_n \mathbin{\backslash\!\backslash} A^+_n \mathbin{/\!\!/} A^+_{n-p-1} = (A^+_n \mathbin{\backslash\!\backslash} A^+_n) \mathbin{/\!\!/} A^+_{n-p-1} \simeq * \mathbin{/\!\!/} A^+_{n-p-1} = BA^+_{n-p-1}.$$

The central equality is due to the face that the double homotopy quotient is the geometric realisation of a bi-simplicial-space and therefore we can take the realisation in either direction first. The final homotopy equivalence is given by Lemma 4.5.18, and the map is given by the projection as required. Finally $* \mathbin{/\!\!/} A^+_{n-p-1}$ is a model for BA^+_{n-p-1} by Lemma 4.5.17. ☐

To address Point 3: there is a map from the geometric realisation of $\mathcal{A}^n_\bullet$ to the classifying space BA^+_n, in Section 6.2 we need to define a map $\|\phi_\bullet\|$ as follows

$$\|\mathcal{A}^n_\bullet\| \xrightarrow{\|\phi_\bullet\|} BA^+_n$$

and Point 4: $\|\phi_\bullet\|$ is highly connected, is the topic of Section 6.5.

LEMMA 6.4.7. *We have that $\|\mathcal{A}^n_\bullet\| \simeq A^+_n \mathbin{\backslash\!\backslash} \|\mathcal{C}^n_\bullet\|$.*

PROOF. The face maps in the bar construction $B_\bullet(*, A^+_n, \mathcal{C}^n_p)$ for the homotopy quotient in $\mathcal{A}^n_p = A^+_n \mathbin{\backslash\!\backslash} \mathcal{C}^n_p$ commute with the face maps in $\mathcal{C}^n_\bullet$ and therefore with the face maps of $\mathcal{A}^n_\bullet$. Therefore the two simplicial directions create a bi-semi-simplicial set and one can realise in either direction first. Realising in the $\mathcal{A}^n_\bullet$ direction first, which has face maps induced by those of $\mathcal{C}^n_\bullet$, completes the proof. $\square$

Recall that $A_n^+ \mathbin{/\mkern-6mu/} *$ is a model for BA_n^+. We therefore define $\|\phi_\bullet\|$ as a map from $A_n^+ \mathbin{/\mkern-6mu/} \|\mathcal{C}^n_\bullet\|$ to $A_n^+ \mathbin{/\mkern-6mu/} *$.

Definition 6.4.8. Define $\phi_\bullet$ to be the semi-simplicial map from the bar construction $B_\bullet(*, A_n^+, \|\mathcal{C}^n_\bullet\|)$ to the bar construction $B_\bullet(*, A_n^+, *)$, defined by collapsing $\|\mathcal{C}^n_\bullet\|$ to a point:

$$\begin{aligned} \phi_p : B_p(*, A_n^+, \|\mathcal{C}^n_\bullet\|) &\to B_p(*, A_n^+, *) \\ [a_1, \ldots, a_p]a &\mapsto [a_1, \ldots, a_p]* \end{aligned}$$

where a_i is in A_n^+ for all i, and a is in $\|\mathcal{C}^n_\bullet\|$. Then the geometric realisation $\|\phi_\bullet\|$ maps the homotopy quotient $A_n^+ \mathbin{/\mkern-6mu/} \|\mathcal{C}^n_\bullet\|$ to the homotopy quotient $A_n^+ \mathbin{/\mkern-6mu/} *$.

Proposition 6.4.9. *If $\|\mathcal{C}^n_\bullet\|$ is $(k-1)$-connected then the map $\|\phi_\bullet\|$ is k-connected.*

Proof. From [**21**, Lemma 2.4] we know that a semi-simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ satisfies that $\|f_\bullet\|$ is k-connected if $f_p : X_p \to Y_p$ is $(k-p)$ connected for all $p \geq 0$. The map $\|\phi_\bullet\|$ is defined level-wise as the projection

$$\phi_p : (A_n^+)^p \times \|\mathcal{C}^n_\bullet\| \to (A_n^+)^p.$$

Therefore since $\|\mathcal{C}^n_\bullet\|$ is $(k-1)$-connected it follows that ϕ_p is k-connected and in particular it is $(k-p)$-connected for all $p \geq 0$. It follows that the geometric realisation $\|\phi_\bullet\|$ is k-connected. $\square$

6.5. High connectivity

This section is concerned with the proof of the following theorem

Theorem 6.5.1. *The geometric realisation $\|\mathcal{C}^n_\bullet\|$ of the semi-simplicial set $\mathcal{C}^n_\bullet$ is $(n-2)$ connected for all n, i.e. $\pi_i(\|\mathcal{C}^n_\bullet\|) = 0$ for $0 \leq i \leq n-2$.*

For the remainder of this chapter, we will refer to the geometric realisation of the semi-simplicial set as a *complex*. Note that by this we do not mean simplicial complex.

6.5.2. High connectivity of complex $\|\mathcal{C}^n_\bullet\|$. There is a specific argument, called a *union of chambers argument* that is often used to prove high connectivity of a complex. It is closely related to the notion of *shellability* and so we recall the definition of a shellable complex.

Definition 6.5.3 (see Björner [**8**]). Let K be a simplicial complex. K is called *pure* if the set

$$T = \{\sigma \in K \mid \sigma \text{ is not properly contained in any other simplex } \}$$

satisfies that all simplices σ in the set T are of the same dimension i.e. K is a union of top dimensional simplices. A *shelling* of a pure complex K is then given by a linear ordering on T such that each σ in T intersects with its predecessors in the ordering at a non-empty union top dimensional faces, or facets, of σ. For instance if the ordering of T is given by $T = \{F_0, F_1, F_2, \ldots\}$ then K is shellable if

$$F_j \cap \bigcup_{i=0}^{j-1} F_i$$

is a non-empty union of facets of F_j for all j. A complex K is *shellable* if it is pure and admits a shelling.

LEMMA 6.5.4. *If a complex K is shellable, and its top dimensional simplices are n-dimensional then it follows that K is $(n-1)$-connected.*

PROOF. Consider a shelling of K given by $T = \{F_0, F_1, F_2, \ldots\}$ for T defined as above. Then F_0 is contractible. We build up K by adding one top dimensional simplex at a time, with ordering specified by the shelling. At each stage when we add a simplex F_j we have that the intersection with $\bigcup_{i=0}^{j-1} F_i$ is a non empty union of facets of F_j. If this intersection is not the whole boundary of F_j then it follows that the addition of F_j to $\bigcup_{i=0}^{j-1} F_i$ did not change the homotopy type, i.e. $\bigcup_{i=0}^{j-1} F_i \simeq (\bigcup_{i=0}^{j-1} F_i) \cup F_j$. If on the other hand the intersection of $\bigcup_{i=0}^{j-1} F_i$ is the boundary of F_j, i.e. all the facets of F_j, then the homotopy type may have changed by wedging with a sphere S^n, as the map ∂F_j to $\bigcup_{i=0}^{j-1} F_i$is null-homotopic by induction. Therefore we can conclude that building up the whole complex K changes the homotopy type from the original F_0 by either no change, or the addition of n-spheres. It follows that K is $(n-1)$-connected. ☐

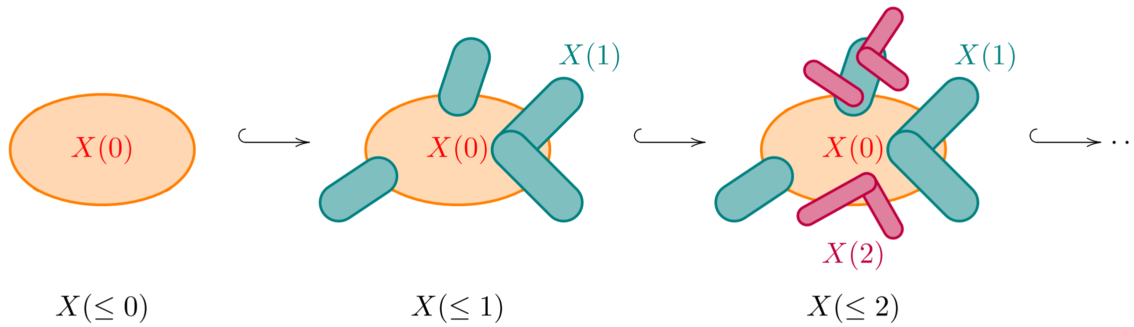
A union of chambers argument applied to a complex X also shows that the complex is highly connected. To follow a union of chambers argument, the complex X must be a union of top dimensional simplices of dimension n for some n, i.e. the complex must be pure, as in Definition 6.5.3. The top dimensional simplices are then called *chambers*. The chambers are ordered, not in a total order but in *batches*, or levels, which we denote $X(k)$ for k in $\mathbb{N}$, such that $X = \bigcup_{k\in\mathbb{N}} X(k)$. Let $X(\leq r)$ be $X = \bigcup_{k=0}^{r} X(k)$. We build X up by adding one batch of chambers at a time, starting at $X(0)$ and adding $X(1)$ to create $X(\leq 1)$, then adding $X(2)$ to $X(\leq 1)$ to create $X(\leq 2)$ and so on.

LEMMA 6.5.5. *Let X, $X(k)$ and $X(\leq k)$ be as above, then X is $(n-1)$-connected if the following three conditions hold*

(1) $X(0)$ is contractible.
(2) For $r \in \mathbb{N}$, all a in $X(r+1)$ satisfy that $a \cap X(\leq r)$ is a non-empty union of top dimensional faces (facets) of a.
(3) If $r \in \mathbb{N}$, and a and b in $X(r+1)$ then $a \cap b$ lies in $X(\leq r)$.

Proof. This proof is similar to the proof of Lemma 6.5.4. We build up X by starting with batch $X(0)$, which by point (1) is contractible. We add batch $X(k)$ to $X(\leq k-1)$ at each stage to get $X(\leq k)$. By point (2) and the proof of Lemma 6.5.4 adding each individual simplex in the batch $X(k)$ either does not change the homotopy type of $X(\leq k-1)$ or changes it by the addition of an n-sphere only. Point (3) tells us that adding on a whole batch of simplices at the same time does not change the homotopy type by anything other than if the addition were of the simplices one at a time. This is because any intersection between the simplices in a batch $X(k)$, takes places in the previous batches $X(\leq k-1)$ where we have already calculated the homotopy.

The diagram below shows a conceptual view of the building up of the complex X, with the cylinders representing chambers, the colours batches and the overlaps intersections.



□

In [**17**], Davis uses a union of chambers argument to prove that the Davis complex Σ_W associated to a Coxeter group is contractible. He does this by showing that the Davis complex is an example of a *basic construction*, which satisfies hypotheses such as those in Lemma 6.5.5. Hepworth's high connectivity results relating to homological stability for Coxeter groups [**31**] also use such an argument. In [**40**], Paris uses a union of chambers argument to show that the universal cover of an analogue of the Salvetti complex for certain Artin monoids is contractible. This proves the $K(\pi, 1)$ conjecture for finite type Artin groups. In this chapter we use a similar union of chambers argument to prove high connectivity. Whilst applying the argument in the case of Artin monoids and the complex we have constructed, numerous technical challenges arise, leading to the proof being split into many separate cases that each have to be approached differently.

To prove high connectivity in our set up we use a union of chambers argument applied to the complex $\|\mathcal{C}^n_\bullet\|$. We filter the top dimensional simplices by the natural numbers as follows:

Definition 6.5.6. For k in $\mathbb{N}$ we define $\mathcal{C}^n(k)$ as follows:

$$\mathcal{C}^n(k) = \bigcup_{\substack{\alpha\in A_n^+,\\ \ell(\alpha)\leq k}} [\![\alpha]\!]_0$$

Where $[\![\alpha]\!]_0$ is the $(n-1)$ simplex in $\|\mathcal{C}^n_\bullet\|$ represented by $[\alpha]_0$ in $\mathcal{C}^n_{n-1}$. Then $\|\mathcal{C}^n_\bullet\|$ is given by $\lim_{k\to\infty}\mathcal{C}^n(k)$.

REMARK 6.5.7. Note that every simplex in $\|\mathcal{C}^n_\bullet\|$ arises as a face of some $[\![\alpha]\!]_0$, since smaller simplices are represented by some $[\![\tau]\!]_k$ for $k>0$ and this is a face of $[\![\tau]\!]_0$. In the language of Definition 6.5.3 $\|\mathcal{C}^n_\bullet\|$ is *pure*.

The union of chambers argument relies on the following two steps:

(A) If $\ell(\alpha)=k+1$ then $[\![\alpha]\!]_0\cap\mathcal{C}^n(k)$ is a non-empty union of top dimensional faces of $[\![\alpha]\!]_0$.
(B) If $\ell(\alpha)=\ell(\beta)=k+1$ then $[\![\alpha]\!]_0\cap[\![\beta]\!]_0\subseteq\mathcal{C}^n(k)$.

which correspond to conditions (2) and (3) in Lemma 6.5.5.

PROPOSITION 6.5.8. *If points (A) and (B) hold then it follows that $\|\mathcal{C}^n_\bullet\|$ is $(n-2)$ connected.*

PROOF. This proof follows from Lemma 6.5.5. We build up $\|\mathcal{C}^n_\bullet\|$ by increasing k in $\mathcal{C}^n(k)$. We start at $\mathcal{C}^n(0)=[\![e]\!]_0$, which is a single simplex and thus contractible, this proves point (1) in Lemma 6.5.5. At each step we build up from $\mathcal{C}^n(k)$ to $\mathcal{C}^n(k+1)$ by adding the set of simplices represented by words in A_n^+ of length $(k+1)$:

$$\bigcup_{\substack{\alpha\in A_n^+,\\ \ell(\alpha)=k+1}}[\![\alpha]\!]_0$$

In the language of Lemma 6.5.5 these are the batches $X(k+1)$ and $X(\leq k)$ is given by $\mathcal{C}^n(k)$. Then point (A) says that when $[\![\alpha]\!]_0$ is added to $\mathcal{C}^n(k)$, the intersection is a non-empty union of facets of $[\![\alpha]\!]_0$. This is precisely point (2) in Lemma 6.5.5, and point (B) is precisely point (3) in Lemma 6.5.5. Therefore the proof follows from the proof of Lemma 6.5.5. ☐

6.5.9. Proof of Point A: Facets of $[\![\alpha]\!]_0$. We first focus on the proof of (A), for which we start with a discussion of the top dimensional faces, or *facets* of a simplex $[\![\alpha]\!]_0$. Consider the face maps

$$\begin{array}{rcl}\partial_q^{n-1}:\mathcal{C}^n_{n-1} & \to & \mathcal{C}^n_{n-2}\\ \partial_q^{n-1}:[\![\alpha]\!]_0 & \mapsto & [\![\alpha\sigma_{2+q-1}\dots\sigma_2]\!]_1\end{array}$$

for $0\leq q\leq n-1$. Here ∂_0^{n-1} is right multiplication by the identity.
Under these face maps the facets of $[\![\alpha]\!]_0$ are given by

$$[\![\alpha]\!]_1,[\![\alpha\sigma_2]\!]_1,[\![\alpha\sigma_3\sigma_2]\!]_1,[\![\alpha\sigma_4\sigma_3\sigma_2]\!]_1,\cdots,[\![\alpha\sigma_n\sigma_{n-1}\dots\sigma_3\sigma_2]\!]_1$$

PROPOSITION 6.5.10. *If $\ell(\alpha)=k+1$, at least one of the facets of $[\![\alpha]\!]_0$ lies in $\mathcal{C}^n(k)$.*

PROOF. We must show that at least one facet of $[\![\alpha]\!]_0$ is also a facet of some simplex $[\![\alpha']\!]_0$, where $\ell(\alpha')\leq k$.

Consider $EndGen_1(\alpha)$. If this is non-empty then there exists η with length at least 1 in A_1^+ such that $\alpha = \alpha'\eta$. It follows that $[\![\alpha]\!]_1 = [\![\alpha'\eta]\!]_1 = [\![\alpha']\!]_1$. Therefore the facet $[\![\alpha]\!]_1$ is also a facet of $[\![\alpha']\!]_0$ and since η had length at least 1 it follows $\ell(\alpha') < \ell(\alpha) = k+1$ and so $[\![\alpha']\!]_0$ is in $\mathcal{C}^n(k)$.

Alternatively if $EndGen_1(\alpha) = \emptyset$, then since $\ell(\alpha) \geq 1$ it follows that $EndGen_n(\alpha) \neq \emptyset$. It follows from these two observations that $\{\sigma_2, \dots \sigma_n\} \cap EndGen_n(\alpha) \neq \emptyset$. For some $2 \leq j \leq n$ we therefore have that $\alpha = \alpha'\sigma_j$. Applying the face map ∂^n_{j-2} gives

$$\begin{aligned} \partial^{n-1}_{j-2}([\![\alpha]\!]_0) &= [\![\alpha\sigma_{j-1}\dots\sigma_2]\!]_1 \\ &= [\![\alpha'\sigma_j\sigma_{j-1}\dots\sigma_2]\!]_1 \\ &= \partial^{n-1}_{j-1}([\![\alpha']\!]_0) \end{aligned}$$

and as before $\ell(\alpha') \leq k$. This shows that the facet $\partial^{n-1}_{j-2}([\![\alpha]\!]_0)$ is also a facet of $[\![\alpha']\!]_0$ and is therefore in $\mathcal{C}^n(k)$. □

To complete the proof of (A) we must show that if a lower dimensional face of $[\![\alpha]\!]_0$ intersects $\mathcal{C}^n(k)$ then this is contained in a top dimensional face, or facet, that intersects $\mathcal{C}^n(k)$. We first describe a general form for faces of $[\![\alpha]\!]_0$.

6.5.11. Proof of Point A: Low dimensional faces of $[\![\alpha]\!]_0$.

Definition 6.5.12. A face of $[\![\alpha]\!]_0$ is obtained by applying a series of face maps to $[\![\alpha]\!]_0$. We denote the series of face maps applied by a tuple $(\partial^{n-1}_{i_2}, \partial^{n-2}_{i_3}, \dots, \partial^{n-r+1}_{i_r})$, and we let $a_j := \sigma_{i_j-1+j}\dots\sigma_j$. That is, the $(j-1)$st map in the tuple corresponds to right multiplication by a_j. We note here that a_j has length i_j and ends with the generator σ_j, unless $i_j = 0$ in which case $a_j = e$.

$$\begin{aligned} \partial^{n-j+1}_{i_j} : \mathcal{C}^n_{n-j+1} &\to \mathcal{C}^n_{n-j} \\ : [\![\alpha]\!]_{j-2} &\mapsto [\![\alpha\sigma_{i_j-1+j}\dots\sigma_j]\!]_{j-1} \\ &= [\![\alpha a_j]\!]_{j-1}. \end{aligned}$$

From now on we assume that the first map in a tuple maps from $\mathcal{C}^n_{n-1}$ to $\mathcal{C}^n_{n-2}$, the second map from $\mathcal{C}^n_{n-2}$ to $\mathcal{C}^n_{n-3}$ and so on. We therefore dispense of the superscripts in the ∂ notation for the face maps when we write these tuples.

With the above notation, an $(n-p-1)$ subsimplex of $[\![\alpha]\!]_0$ occurs when a tuple of face maps $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ is applied to $[\![\alpha]\!]_0$. The image of these maps is then the subsimplex $[\![\alpha a_2 \dots a_{p+1}]\!]_p$ with a_j defined as in Definition 6.5.12 above.

Lemma 6.5.13. *With the above notation, the tuple of face maps $(\partial_{i_j})^{p+1}_{j=2}$ can be organised such that $i_{j+1} \geq i_j$ for all j, which translates to $\ell(a_{j+1}) \geq \ell(a_j)$.*

Proof. This is a direct consequence of Lemma 4.5.7. □

LEMMA 6.5.14. *The $(n-p-1)$ subsimplex of $[\![\alpha]\!]_0$ given by $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ or alternatively $[\![\alpha a_2 \dots a_{p+1}]\!]_p$ is a subsimplex of the following facets of $[\![\alpha]\!]_0$:*

- $\partial_{i_2}([\![\alpha]\!]_0) = [\![\alpha a_2]\!]_1$
- $\partial_{i_3+1}([\![\alpha]\!]_0) = [\![\alpha a_3 \sigma_2]\!]_1$
- $\partial_{i_4+2}([\![\alpha]\!]_0) = [\![\alpha a_4 \sigma_3 \sigma_2]\!]_1$
- $\dots$
- $\partial_{i_{p+1}+p-1}([\![\alpha]\!]_0) = [\![\alpha a_{p+1} \sigma_p \dots \sigma_2]\!]_1$

In general the face map $\partial_{i_j+(j-2)}$ acts on $[\![\alpha]\!]_0$ to give $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1$.

PROOF. It is enough to show that $\partial_{i_j+(j-2)}$ can act as the first face map in the tuple $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ for all j. Recall from Lemma 6.5.13 that in the tuple we have $i_{j+1} \geq i_j$ for all j. It therefore follows that using the simplicial identities, the tuple can be rearranged to $(\partial_{i_j+(j-2)}, \partial_{i_2}, \partial_{i_3}, \dots, \widehat{\partial_{i_j}}, \dots, \partial_{i_{p+1}})$. □

For the remainder of this section, let α in A_n^+ with $\ell(\alpha) = k+1$. The aim of this section is to show that if the $(n-p-1)$ subsimplex of $[\![\alpha]\!]_0$ given by $(\partial_{i_2}, \partial_{i_3}, \dots, \partial_{i_{p+1}})$ or alternatively $[\![\alpha a_2 \dots a_{p+1}]\!]_p$ is in $\mathcal{C}^n(k)$ then it follows one of the facets of $[\![\alpha]\!]_0$ from Lemma 6.5.14 is also in $\mathcal{C}^n(k)$. The proof of (A) will then follow.

DEFINITION 6.5.15. If $[\![\alpha a_2 \dots a_{p+1}]\!]_p$ is in $\mathcal{C}^n(k)$ then it is also a $(n-p-1)$ subsimplex of a simplex $[\![\beta]\!]_0$ for some β in A_n^+ such that $\ell(\beta) \leq k$. The subsimplex is therefore obtained from $[\![\beta]\!]_0$ by applying a tuple of face maps, denote these $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$ and order as in Lemma 6.5.13 such that $l_{j+1} \geq l_j$ for all j. Define $b_j := \sigma_{l_j-1+j} \dots \sigma_j$ and when $l_j = 0$ let $b_j = e$. Then $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$ applied to $[\![\beta]\!]_0$ gives the $(n-p-1)$ simplex $[\![\beta b_2 \dots b_{p+1}]\!]_p$. By construction $[\![\beta b_2 \dots b_{p+1}]\!]_p = [\![\alpha a_2 \dots a_{p+1}]\!]_p$. We recall here that $\ell(a_j) = i_j$ and $\ell(b_j) = l_j$.

LEMMA 6.5.16. *We choose β and b_j as defined above, such that $\sum_{k=2}^{p+1} l_k$ is minimal, corresponding to $b_2 \dots b_{p+1}$ being of minimal length. This choice of $b_2 \dots b_{p+1}$ then corresponds to either:*

$$[\![\alpha a_2 \dots a_{p+1}]\!]_p = [\![\beta]\!]_p \text{ that is, } l_j = 0 \,\forall\, j$$

or

$$\ell(\beta) = \ell(\alpha) - 1 = k.$$

PROOF. Suppose that β and b_j are chosen such that $\sum_{k=2}^{p+1} l_k$ is minimal, and furthermore suppose that $\ell(\beta) < \ell(\alpha) - 1$ and $\sum_{k=2}^{p+1} l_k > 0$. Then some $l_k \neq 0$: set j to be minimal such that $l_j \neq 0$. Then $b_j = \sigma_{l_j-1+j} \dots \sigma_j \neq e$ and $[\![\beta b_2 \dots b_{p+1}]\!]_p = [\![\beta b_j \dots b_{p+1}]\!]_p = [\![\beta \sigma_{l_j-1+j} \dots \sigma_j b_{j+1} \dots b_{p+1}]\!]_p$. But this is the tuple of face maps $(\partial_{l_j-1}, \partial_{l_{j+1}}, \dots, \partial_{l_{p+1}})$ applied to $[\![\beta \sigma_{l_j-1+j}]\!]_0$. Since $\ell(\beta) < \ell(\alpha) - 1$ it follows that $\ell(\beta \sigma_{l_j-1+j}) \leq \ell(\alpha) - 1$ and so $[\![\beta \sigma_{l_j-1+j}]\!]_0$ is in $\mathcal{C}^n(k)$. However the tuple for $\beta \sigma_{l_j-1+j}$ has the sum of its corresponding l_j less than the original tuple for β. This is a contradiction, as β was chosen to have minimal $\sum_{k=2}^{p+1} l_k$. Therefore either $\sum_{k=2}^{p+1} l_k = 0$, or alternatively $\ell(\beta) = \ell(\alpha) - 1$. □

For the remainder of this proof, assume β and b_j are chosen such that $\sum_{k=2}^{p+1} l_k$ is minimal, so we have

$$[\![\beta b_2 \dots b_{p+1}]\!]_p = [\![\alpha a_2 \dots a_{p+1}]\!]_p$$

for either $\sum_{k=2}^{p+1} l_k = 0$ or $\ell(\beta) = \ell(\alpha) - 1 = k$. We use the following notation throughout the remainder of this chapter.

Definition 6.5.17. Let $a := a_2 \dots a_{p+1}$ and $b := b_2 \dots b_{p+1}$. Note that $\sum_{k=2}^{p+1} l_k = 0$ corresponds to $b = e$. So we have

$$[\![\alpha a]\!]_p = [\![\beta b]\!]_p$$

and we recall that this is equivalent to $\overline{\alpha a} = \overline{\beta b}$ in $A^+(n;p)$. Let $\gamma := \overline{\alpha a} = \overline{\beta b}$, and define u and v in A_p^+ such that

$$\alpha a = \gamma u \text{ and } \beta b = \gamma v.$$

We complete the proof of (A) by splitting into three cases:

(i) $\ell(\beta b) < \ell(\alpha a)$
(ii) $\ell(\beta b) = \ell(\alpha a)$
(iii) $\ell(\beta b) > \ell(\alpha a)$

and since multiplication in the Artin monoid corresponds to adding lengths the conditions of these cases correspond to analogous conditions on the lengths of u and v.

Remark 6.5.18. Note that if $\sum_{k=2}^{p+1} l_k = 0$ then $b = e$, and since $\ell(\beta) < \ell(\alpha)$ it follows we are therefore in case (i):$\ell(\beta b) < \ell(\alpha a)$.

We prove the three cases one by one in the following subsections. This involves many technical lemmas, and in particular computation of least common multiples of strings of words. We therefore include all these technical lemmas on least common multiples in a separate section and refer to them as required.

6.5.19. Proof of Point A: least common multiple calculations. Recall from Definition 6.5.12 that a face of $[\![\alpha]\!]_0$ is obtained by applying a series of face maps to $[\![\alpha]\!]_0$. We denote the series of face maps applied by a tuple $(\partial_{i_2}^{n-1}, \partial_{i_3}^{n-2}, \dots, \partial_{i_r}^{n-r+1})$, and we let $a_j = \sigma_{i_j-1+j} \dots \sigma_j$ and when $i_j = 0$ let $a_j = e$. That is, the $(j-1)$st map in the tuple corresponds to right multiplication by a_j. We let $a = a_2 \dots a_{p+1}$. Recall also that if $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$ then the subsimplex is also obtained from some $[\![\beta]\!]_0$ for $\ell(\beta) \leq k$, by applying a tuple of face maps $(\partial_{l_2}, \partial_{l_3}, \dots, \partial_{l_{p+1}})$. Recall $b_j := \sigma_{l_j-1+j} \dots \sigma_j$ and when $l_j = 0$ let $b_j = e$. Let $b = b_2 \dots b_{p+1}$. By construction $[\![\beta b]\!]_p = [\![\alpha a]\!]_p$. Recall from Definition 4.2.8 that for α and β two words in A^+, we denote the least common multiple of α and β (if it exists) by $\Delta(\alpha, \beta)$.

Lemma 6.5.20. *With notation as above,* $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$.

Proof. We must show

(a) $a_{j+1} \preceq_R a_{j+1}\sigma_j a_{j+1}$ and $\sigma_j \preceq_R a_{j+1}\sigma_j a_{j+1}$.
(b) if x in A_n^+ is a common multiple of a_{j+1} and σ_j, then $a_{j+1}\sigma_j a_{j+1} \preceq_R x$.

Recall $a_{j+1} := \sigma_{i_{j+1}+j} \dots \sigma_{j+1}$. Without loss of generality, relabel $j = 1$ and $i_{j+1} + j = k$. Then $a_{j+1} = \sigma_k \dots \sigma_2$ and $\sigma_j = \sigma_1$

To prove (a) we note $a_{j+1} \preceq_R a_{j+1}\sigma_j a_{j+1}$ from observation, and also

$$\begin{aligned}
a_{j+1}\sigma_j a_{j+1} &= (\sigma_k \dots \sigma_2)\sigma_1(\sigma_k \dots \sigma_2) \\
&= ((\sigma_k\sigma_{k-1}\sigma_k)\sigma_{k-2} \dots \sigma_2)\sigma_1(\sigma_{k-1} \dots \sigma_2) \\
&= ((\sigma_{k-1}\sigma_k\sigma_{k-1})\sigma_{k-2} \dots \sigma_2)\sigma_1(\sigma_{k-1} \dots \sigma_2) \\
&= (\sigma_{k-1}\sigma_k(\sigma_{k-1}\sigma_{k-2}\sigma_{k-1})\sigma_{k-3} \dots \sigma_2)\sigma_1(\sigma_{k-2} \dots \sigma_2) \\
&= (\sigma_{k-1}\sigma_k(\sigma_{k-2}\sigma_{k-1}\sigma_{k-2})\sigma_{k-3} \dots \sigma_2)\sigma_1(\sigma_{k-2} \dots \sigma_2) \\
&= \cdots \\
&= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3\sigma_2)\sigma_1(\sigma_2) \\
&= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3)(\sigma_2\sigma_1\sigma_2) \\
&= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_1)
\end{aligned}$$

so $\sigma_1 = \sigma_j \preceq_R a_{j+1}\sigma_j a_{j+1}$.

To prove (b) we note that $a_{j+1}\sigma_j a_{j+1}$ is a common multiple, and we show by induction on $\ell(a_{j+1})$ that any common multiple x must satisfy $a_{j+1}\sigma_j a_{j+1} \preceq_R x$. When $\ell(a_{j+1}) = 1$, $a_{j+1} = \sigma_2$ and we have $\Delta(\sigma_2, \sigma_1) = \sigma_2\sigma_1\sigma_2 = a_{j+1}\sigma_j a_{j+1}$. When $\ell(a_{j+1}) = r - 1$ for $r \geq 2$, assume that $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$ and prove for $\ell(a_{j+1}) = r$. Assume x satisfies $a_{j+1} \preceq_R x$ and $\sigma_j \preceq_R x$. Since $\ell(a_{j+1}) = r$ this means $a_{j+1} = \sigma_{r+1} \dots \sigma_2$ and so $\sigma_{r+1} \dots \sigma_2 \preceq_R x$ which in particular gives $\sigma_r \dots \sigma_2 \preceq_R x$. By the inductive hypothesis it follows that $\Delta(\sigma_r \dots \sigma_2, \sigma_1) = (\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2)$ and this is in $EndMon_n(x)$ by Lemma 4.3.9. Let $x = x'(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2)$. Then since $\sigma_{r+1} \dots \sigma_2 \preceq_R x$, by cancellation of $\sigma_r \dots \sigma_2$ we have that $\sigma_{r+1} \preceq x'(\sigma_r \dots \sigma_2)\sigma_1 = x'\sigma_r(\sigma_{r-1} \dots \sigma_2\sigma_1)$. Since σ_{r+1} letterwise commutes with $(\sigma_{r-1} \dots \sigma_2\sigma_1)$, from Lemma 4.3.12 $\sigma_{r+1} \preceq_R x'\sigma_r$. From Lemma 4.3.9 it follows $\Delta(\sigma_{r+1}, \sigma_r) = \sigma_r\sigma_{r+1}\sigma_r \preceq_R x'\sigma_r$. By cancellation of σ_r this gives $x' = x''\sigma_r\sigma_{r+1}$, so

$$\begin{aligned}
x &= (x')(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= (x''\sigma_r\sigma_{r+1})(\sigma_r \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= x''(\sigma_r\sigma_{r+1}\sigma_r)(\sigma_{r-1} \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= x''(\sigma_{r+1}\sigma_r\sigma_{r+1})(\sigma_{r-1} \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= x''(\sigma_{r+1}\sigma_r\sigma_{r+1}\sigma_{r-1} \dots \sigma_2)\sigma_1(\sigma_r \dots \sigma_2) \\
&= x''(\sigma_{r+1}\sigma_r\sigma_{r-1} \dots \sigma_2)\sigma_1(\sigma_{r+1}\sigma_r \dots \sigma_2) \\
&= x''a_{j+1}\sigma_j a_{j+1}
\end{aligned}$$

as required. $\square$

LEMMA 6.5.21. *Recall from Lemma 6.5.20 that* $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$. *We have that*

$$a_{j+1}\sigma_j a_{j+1} = \hat{a}_j a_j a_{j+1}\sigma_j$$

where $\hat{a}_j = \sigma_{i_{j+1}+j-1} \dots \sigma_{i_j+j}$ *and letterwise commutes with* $a_2 \dots a_{j-1}$.

PROOF. Recall $a_{j+1} := \sigma_{i_{j+1}+j} \dots \sigma_{j+1}$ and $a_j := \sigma_{i_j-1+j} \dots \sigma_j$. Without loss of generality, relabel $j = 1$ and $i_{j+1} + j = k$, and $i_j - 1 + j = l$. Then $a_{j+1} = \sigma_k \dots \sigma_2$ and $\sigma_j = \sigma_1$, and $a_j = \sigma_l \dots \sigma_1$. Note that since $i_{j+1} \geq i_j$ then $k > l$. We want to show that $a_{j+1}\sigma_j a_{j+1} = \hat{a}_j a_j a_{j+1}\sigma_j$ where $\hat{a}_j = \sigma_{k-1} \dots \sigma_{l+1}$.

Now recall from the proof of Lemma 6.5.20 that

$$\begin{aligned} a_{j+1}\sigma_j a_{j+1} &= (\sigma_k \dots \sigma_2)\sigma_1(\sigma_k \dots \sigma_2) \\ &= \cdots \\ &= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_1). \end{aligned}$$

We now move all generators in this expression as far to left as possible, past all other generators that they commute with.

$$\begin{aligned} a_{j+1}\sigma_j a_{j+1} &= (\sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_1) \\ &= \sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_3\sigma_4\sigma_2(\sigma_3\sigma_1)\sigma_2\sigma_1 \\ &= \sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_3\sigma_4\sigma_2(\sigma_1\sigma_3)\sigma_2\sigma_1 \\ &= \sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_3(\sigma_4\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) \\ &= \sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots \sigma_3(\sigma_2\sigma_1\sigma_4)(\sigma_3\sigma_2\sigma_1) \\ &= \sigma_{k-1}\sigma_k\sigma_{k-2}\sigma_{k-1} \dots (\sigma_5\sigma_3\sigma_2\sigma_1)(\sigma_4\sigma_3\sigma_2\sigma_1) \\ &= \cdots \\ &= \sigma_{k-1}(\sigma_k\sigma_{k-2}\sigma_{k-3} \dots \sigma_2\sigma_1)(\sigma_{k-1}\sigma_{k-2} \dots \sigma_2\sigma_1) \\ &= \sigma_{k-1}(\sigma_{k-2}\sigma_{k-3} \dots \sigma_2\sigma_1\sigma_k)(\sigma_{k-1}\sigma_{k-2} \dots \sigma_2\sigma_1) \\ &= (\sigma_{k-1}\sigma_{k-2}\sigma_{k-3} \dots \sigma_2\sigma_1)(\sigma_k\sigma_{k-1}\sigma_{k-2} \dots \sigma_2\sigma_1) \\ &= ((\sigma_{k-1}\sigma_{k-2} \dots \sigma_{l+1})(\sigma_l \dots \sigma_2\sigma_1))((\sigma_k\sigma_{k-1}\sigma_{k-2} \dots \sigma_2)\sigma_1) \\ &= (\sigma_{k-1}\sigma_{k-2} \dots \sigma_{l+1})(\sigma_l \dots \sigma_2\sigma_1)(\sigma_k\sigma_{k-1}\sigma_{k-2} \dots \sigma_2)(\sigma_1) \\ &= (\hat{a}_j)(a_j)(a_{j+1})(\sigma_j). \end{aligned}$$

Then $\hat{a}_j = \sigma_{k-1} \dots \sigma_{l+1}$ where l is the maximal index of a generator appearing in a_j. Since $i_j \geq i_{j-1}$ it follows that $l-1$ is the maximal index of a generator appearing in a_{j-1} and hence in the string $a_2 \dots a_{j-1}$. Therefore $\hat{a}_j$ letterwise commutes with $a_2 \dots a_{j-1}$ since the indices of the generators in each word pairwise differ by at least two.

$\square$

Definition 6.5.22. Recall the definition of a_j and b_j for $2 \le j \le p+1$. Define c_j as follows

$$c_j = \begin{cases} a_j & \text{if } \ell(a_j) \ge \ell(b_j) \\ b_j & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

for $2 \le j \le p+1$. Define $c := c_2 \dots c_{p+1}$.

Lemma 6.5.23. *With c as defined in 6.5.22 and a and b as defined in 6.5.17 we have $c = \Delta(a, b)$.*

Proof. We must prove that

(a) $a \preceq_R c$ and $b \preceq_R c$, i.e. there exist a' and b' such that $c = a'a = b'b$
(b) if x in A_n^+ is a common multiple of a and b, then $c \preceq_R x$.

To prove (a), we show that $c = a'a$, and the proof that $c = b'b$ is symmetric. We have that

$$c_j = a'_j a_j \text{ where } a'_j = \begin{cases} e & \text{if } \ell(a_j) \ge \ell(b_j) \\ \sigma_{l_j+j-1} \dots \sigma_{i_j+j} & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

The smallest generator index in a'_j is $(i_j + j)$ and the largest generator index in $a_2 \dots a_{j-1}$ is $(i_{j-1} + (j-1) - 1)$. Therefore, since $\mid (i_j + j) - (i_{j-1} + (j-1) - 1) \mid = \mid (i_j - i_{j-1}) + 2) \mid \ge 2$, since $i_j \ge i_{j-1}$, a'_j letterwise commutes with $a_2 \dots a_{j-1}$. Let $a' = a'_2 \dots a'_{p+1}$. It follows

$$\begin{aligned} c &= c_2 \dots c_{p+1} \\ &= (a'_2 a_2)(a'_3 a_3) \dots (a'_{p+1} a_{p+1}) \\ &= a'_2 a'_3 a_2 a_3 \dots (a'_{p+1} a_{p+1}) \\ &= a'_2 a'_3 \dots a'_{p+1} a_2 a_3 \dots a_{p+1} \\ &= (a'_2 a'_3 \dots a'_{p+1})(a_2 a_3 \dots a_{p+1}) \\ &= a'a \end{aligned}$$

which completes the proof of (a).

To prove (b) assume x is a common multiple of a and b.

Claim: If $c_k \dots c_{p+1} \preceq_R x$ for some $2 \le k \le p+1$ then $x = x_k c_k \dots c_{p+1}$ for some x_k in A_n^+. We claim that x_k satisfies $a_2 \dots a_{k-1} \preceq_R x_k$ and $b_2 \dots b_{k-1} \preceq_R x_k$.

Given the claim, the proof of (b) will follow since $a = (a_2 \dots a_{p+1}) \preceq_R x$ and $b = (b_2 \dots b_{p+1}) \preceq_R x$ implies that $c_{p+1} \preceq_R x$, so $x = x_{p+1} c_{p+1}$. But then x_{p+1} satisfies $a_2 \dots a_p \preceq_R x_{p+1}$ and $b_2 \dots b_p \preceq_R x_{p+1}$ by the claim. In particular this means $c_p \preceq_R x_{p+1}$ and it follows that $x = x_p c_p c_{p+1}$. Continuing in this manner we arrive at $x = x_2(c_2 \dots c_{p+1}) = x_2 c$ and so $c \preceq_R x$. It therefore remains to prove the claim.

Since $c_k \dots c_{p+1} = (a'_k a_k) \dots (a'_{p+1} a_{p+1}) = (a'_k \dots a'_{p+1})(a_k \dots a_{p+1})$ it follows that

$$\begin{array}{rcl} x & = & x_k(c_k \dots c_{p+1}) \\ & = & x_k(a'_k \dots a'_{p+1})(a_k \dots a_{p+1}) \\ & = & y_k(a_k \dots a_{p+1}) \text{ for } y_k = x_k(a'_k \dots a'_{p+1}). \end{array}$$

Since x is a common multiple of a and b then we also have $a = (a_2 \dots a_{p+1}) \preceq_R x$, i.e for some z_k.

$$x = z_k(a_2 \dots a_{p+1})$$

Therefore by cancellation of $(a_k \dots a_{p+1})$,

$$y_k = z_k(a_2 \dots a_{k-1})$$

By Lemma 4.3.11, $\Delta((a'_k \dots a'_{p+1}), (a_2 \dots a_{k-1})) \preceq_R y_k$. Since the two words letterwise commute $\Delta((a'_k \dots a'_{p+1}), (a_2 \dots a_{k-1})) = (a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})$ and so

$$y_k = w_k(a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})$$

for some w_k in A_n^+. So we have

$$\begin{array}{rcl} x & = & x_k(c_k \dots c_{p+1}) \\ & = & y_k(a_k \dots a_{p+1}) \\ & = & w_k(a_2 \dots a_{k-1})(a'_k \dots a'_{p+1})(a_k \dots a_{p+1}) \\ & = & w_k(a_2 \dots a_{k-1})((a'_k \dots a'_{p+1})(a_k \dots a_{p+1})) \\ & = & w_k(a_2 \dots a_{k-1})(c_k \dots c_{p+1}) \end{array}$$

and by cancellation of $c_k \dots c_{p+1}$ on the first and final lines of the above equation, we have that $(a_2 \dots a_{k-1}) \preceq_R x_k$ as required. The proof for $(b_2 \dots b_{k-1}) \preceq_R x_k$ is symmetrical. This completes the proof of the Claim and thus of (b). □

6.5.24. Proof of Point A: Proof of case (i):$\ell(\beta b) < \ell(\alpha a)$.

PROPOSITION 6.5.25. *Under the hypotheses of case (i), it follows that $EndGen_p(\alpha a) \neq \emptyset$.*

PROOF. Recall that for some u and v in A_p^+, $\alpha a = \gamma u$ and $\beta b = \gamma v$. If $\ell(\beta b) < \ell(\alpha a)$ then it follows $\ell(\gamma v) < \ell(\gamma u)$ and consequently $\ell(v) < \ell(u)$, since multiplication in A_n^+ corresponds to addition of lengths. Since the inequality is strict, it follows that $\ell(u) \neq 0$, i.e. $u \neq e$. It follows that since $\alpha a = \gamma u$, $u \in EndMon_p(\alpha a)$ so in particular $EndGen_p(\alpha a) \neq \emptyset$. □

REMARK 6.5.26. To prove point (A) in the setting of case (i), it is therefore enough to prove that if $EndGen_p(\alpha a) \neq \emptyset$ and $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$ that a facet containing $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$.

PROPOSITION 6.5.27. *If $EndGen_0(\alpha a) \neq \emptyset$ then the facet $[\![\alpha a_2]\!]_1$ containing $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$.*

PROOF. Consider τ in $EndGen_0(\alpha a)$. Then since the generators S_0 of A_0^+ commute with $\sigma_2, \ldots, \sigma_n$ it follows that τ letterwise commutes (see Definition 4.3.10) with a since $a = a_2 \ldots a_{p+1}$ and therefore a only contains letters in the set of generators $\{\sigma_2, \ldots \sigma_n\}$. We therefore have that τ and a are both in $EndMon_n(\alpha a)$ and they letterwise commute. It follows from Lemma 4.3.12 that τ is in $EndMon_n(\alpha)$, so for some α' in A_n^+, $\alpha = \alpha'\tau$ with $\ell(\alpha') < \ell(\alpha)$.

The facet $[\![\alpha a_2]\!]_1$ therefore satisfies

$$[\![\alpha a_2]\!]_1 = [\![\alpha'\tau a_2]\!]_1 = [\![\alpha' a_2 \tau]\!]_1 = [\![\alpha' a_2]\!]_1.$$

Here the final equality is due to $\overline{\alpha' a_2 \tau} = \overline{\alpha' a_2}$ where the reduction is taken with respect to A_1^+ (from Lemma 4.3.4). The penultimate equality is due to the fact τ and a_2 letterwise commute. Since $\ell(\alpha') < \ell(\alpha)$, $[\![\alpha']\!]_0$ is in $\mathcal{C}^n(k)$ and $[\![\alpha' a_2]\!]_1$ is a facet of $[\![\alpha']\!]_0$. Therefore $[\![\alpha a_2]\!]_1$ is in $\mathcal{C}^n(k)$ and this completes the proof. $\square$

The case where $EndGen_p(\alpha a) \neq \emptyset$ but $EndGen_0(\alpha a) = \emptyset$ requires the following technical lemma.

LEMMA 6.5.28. *Suppose $a_j \neq e$, then the words a_j and a_{j+1} as defined in Definition 6.5.12 satisfy $a_{j+1}\sigma_j = \bar{a}_j a_j$, for some $\bar{a}_j$ in A_n^+ with $\ell(\bar{a}_j) \geq 1$, since $\ell(a_{j+1}) \geq \ell(a_j) \geq 1$. Furthermore $\bar{a}_j$ letterwise commutes with $a_2 \ldots a_{j-1}$. Regardless of whether or not $a_j = e$, $a_{j+1}\sigma_j$ corresponds to the face map $\partial^{n-j+1}_{i_{j+1}+1}$.*

PROOF. If $a_j \neq e$ then $a_{j+1}\sigma_j = \bar{a}_j a_j$. That is

$$\begin{aligned} a_{j+1}\sigma_j &= (\sigma_{i_{j+1}+j} \ldots \sigma_{j+1})\sigma_j \\ &= (\sigma_{i_{j+1}+j} \ldots \sigma_{i_j+j})(\sigma_{i_j+j-1} \ldots \sigma_{j+1})\sigma_j \\ &= (\sigma_{i_{j+1}+j} \ldots \sigma_{i_j+j})(\sigma_{i_j+j-1} \ldots \sigma_{j+1}\sigma_j) \\ &= (\sigma_{i_{j+1}+j} \ldots \sigma_{i_j+j})a_j \\ &= \bar{a}_j a_j \end{aligned}$$

so $\bar{a}_j = \sigma_{i_{j+1}+j} \ldots \sigma_{i_j+j}$. The letters appearing in $a_2 \ldots a_{j-1}$ are $\{\sigma_2, \ldots, \sigma_{i_{j-1}+(j-1)-1}\}$ and so to prove that $\bar{a}_j$ letterwise commutes with $a_2 \ldots a_{j-1}$ it is enough to show that the set $A = \{\sigma_{i_j+j}, \ldots, \sigma_{i_{j+1}+j}\}$ pairwise commutes with the set $B = \{\sigma_2, \ldots, \sigma_{i_{j-1}+(j-1)-1}\}$. The largest index of a generator in B is $i_{j-1} + (j-1) - 1$ and the smallest index of a generator in A is $i_j + j$ so it is enough to show $\mid (i_j + j) - (i_{j-1} + (j-1) - 1) \mid = \mid (i_j - i_{j-1}) + 2 \mid \geq 2$. This holds since $i_j \geq i_{j-1}$, and so $\bar{a}_j$ and $a_2 \ldots a_{j-1}$ letterwise commute. Regardless of whether or not $a_j = e$, $a_{j+1}\sigma_j = \bar{a}_j a_j = \sigma_{i_{j+1}+j} \ldots \sigma_j$ corresponds to the face map $\partial^{n-j+1}_{i_{j+1}+1}$ as in Definition 6.5.12. $\square$

PROPOSITION 6.5.29. *If $EndGen_p(\alpha a) \neq \emptyset$ but $EndGen_0(\alpha a) = \emptyset$ then some σ_j is in $EndGen_p(\alpha a)$ for $1 \leq j \leq p$. Then the facet $[\![\alpha a_j \sigma_{j-1} \ldots \sigma_2]\!]_1$ containing $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$.*

Proof. If $EndGen_0(\alpha a) = \emptyset$ and $EndGen_p(\alpha a) \neq \emptyset$ it follows that $\{\sigma_1, \sigma_2, \dots \sigma_p\} \cap EndGen_p(\alpha a) \neq \emptyset$, so some σ_j is in $EndGen_p(\alpha a)$ for $1 \leq j \leq p$. We have that σ_j and $a = a_2 \dots a_{p+1}$ are both in $EndMon_n(\alpha a)$. In particular σ_j and $a_{j+2} \dots a_{p+1}$ are both in $EndMon_n(\alpha a)$. Since σ_j and $a_{j+2} \dots a_{p+1}$ letterwise commute we have from Lemma 4.3.12 that σ_j is in $EndMon_n(\alpha a_2 \dots a_{j+1})$. Since a_{j+1} is also in $EndMon_n(\alpha a_2 \dots a_{j+1})$, from Lemma 4.3.9 we have $\Delta(a_{j+1}, \sigma_j)$ is in $EndMon_n(\alpha a_2 \dots a_{j+1})$. From Lemma 6.5.20 we have $\Delta(a_{j+1}, \sigma_j) = a_{j+1}\sigma_j a_{j+1}$ so $a_{j+1}\sigma_j a_{j+1}$ is in $EndMon_n(\alpha a_2 \dots a_{j+1})$. By cancellation of a_{j+1} it follows that $a_{j+1}\sigma_j$ is in $EndMon_n(\alpha a_2 \dots a_j)$, so $\alpha a_2 \dots a_j = \alpha'(a_{j+1}\sigma_j)$ for some α' in A_n^+.

Recall Lemma 6.5.28 and split into two cases:

(a) $a_j \neq e$

(b) $a_2 = \dots = a_j = e$

For case (a) recall from Lemma 6.5.28 that $a_{j+1}\sigma_j = \bar{a}_j a_j$ and $\bar{a}_j$ letterwise commutes with $a_2 \dots a_{j-1}$. Together with $\alpha a_2 \dots a_j = \alpha'(a_{j+1}\sigma_j)$ this gives

$$\begin{aligned} \alpha a_2 \dots a_j &= \alpha'(a_{j+1}\sigma_j) \\ &= \alpha'(\bar{a}_j a_j) \\ \Rightarrow \alpha a_2 \dots a_{j-1} &= \alpha' \bar{a}_j \text{ by cancellation of } a_j \end{aligned}$$

Now $\alpha(a_2 \dots a_{j-1}) = \alpha'\bar{a}_j$ and $\bar{a}_j$ letterwise commutes with $a_2 \dots a_{j-1}$. By Lemma 4.3.12 it follows that $\bar{a}_j$ is in $EndMon_n(\alpha)$, that is there exists α'' in A_n^+ such that $\alpha = \alpha''\bar{a}_j$.

Then the facet $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1$ satisfies

$$\begin{aligned} & [\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1 \\ = \; & [\![\alpha'' \bar{a}_j a_j \sigma_{j-1} \dots \sigma_2]\!]_1 \end{aligned}$$

and by Lemma 6.5.28 $\bar{a}_j a_j$ is a face map $\partial^{n-j+1}_{i_{j+1}+1}$, so $\bar{a}_j a_j \sigma_{j-1} \dots \sigma_2$ is also a face map $\partial^{n-1}_{i_{j+1}+j-1}$, and therefore $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1$ is also a facet of $[\![\alpha'']\!]_0$. Since $\ell(\bar{a}_j) \geq 1$ it follows $\ell(\alpha'') < \ell(\alpha)$ and so $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1 \in \mathcal{C}^n(k)$.

For case (b), $a_2 = \dots = a_j = e$ gives that $a_{j+1}\sigma_j$ is in $EndMon_n(\alpha)$, so $\alpha = \alpha' a_{j+1}\sigma_j$ for some α' in A_n^+ with $\ell(\alpha') < \ell(\alpha)$. Then the facet $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1$ satisfies

$$\begin{aligned} & [\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1 \\ = \; & [\![(\alpha' a_{j+1}\sigma_j) a_j \sigma_{j-1} \dots \sigma_2]\!]_1 \\ = \; & [\![\alpha'(a_{j+1}\sigma_j \sigma_{j-1} \dots \sigma_2)]\!]_1 \text{ since } a_j = e \end{aligned}$$

and as before by Lemma 6.5.28 this is a face of $[\![\alpha']\!]_0$ which is in $\mathcal{C}^n(k)$ as required. ☐

This concludes the proof of case (i).

6.5.30. Proof of Point A: Proof of case (ii):$\ell(\beta b) = \ell(\alpha a)$.

Proposition 6.5.31. *Recall that for some u and v in A_p^+, and γ in A_n^+ with $EndMon_p(\gamma) = \emptyset$, that $\alpha a = \gamma u$ and $\beta b = \gamma v$. If we are in case (ii) then we only need to consider when $\alpha a = \beta b = \gamma$.*

Proof. Case (ii) states that $\ell(\beta b) = \ell(\alpha a)$, which implies that $\ell(\gamma u) = \ell(\gamma v)$ which in turn implies $\ell(u) = \ell(v)$ by cancellation. If $u \neq e$ then αa satisfies $EndGen_p(\alpha a) \neq \emptyset$. By Remark 6.5.26 it was under this hypothesis that we proved case (i), i.e. if this holds then we have proved in the proof of case (i) that a facet containing $[\![\alpha a]\!]_p$ lies in $\mathcal{C}^n(k)$. Therefore we can assume $u = e$, which implies $v = e$ since they have the same length. Therefore $\alpha a = \beta b = \gamma$. □

Recall the definition of c_j as in Definition 6.5.22:

$$c_j = \begin{cases} a_j & \text{if } \ell(a_j) \geq \ell(b_j) \\ b_j & \text{if } \ell(a_j) < \ell(b_j) \end{cases}$$

for $2 \leq j \leq p+1$. Recall $c = c_2 \dots c_{p+1}$. Recall that since $\ell(\beta) < \ell(\alpha)$ then in case (ii): $\ell(\beta b) = \ell(\alpha a)$ that it follows $\ell(b) > \ell(a)$.

Proposition 6.5.32. *With the notation as above, there exists at least one j for which $c_j = b_j \neq a_j$. Consider the maximum j for which $c_j = b_j \neq a_j$. Then the facet $[\![\alpha a_j \sigma_{j-1} \dots \sigma_2]\!]_1$ of $[\![\alpha]\!]_0$ containing $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$.*

Proof. Recall $c = a'a = b'b$ where $a' = a'_2 \dots a'_{p+1}$ and $b' = b'_2 \dots b'_{p+1}$ as defined in the proof of Lemma 6.5.23. We fist prove the existence of j in the statement. Note since $\ell(\beta) < \ell(\alpha)$ it follows that $b \neq e$ and so from Lemma 6.5.16 it follows that $\ell(\beta) = \ell(\alpha) - 1$ which gives $\ell(b) = \ell(a) + 1$. Putting this together we get $c = a'a = b'b$ and $\ell(b) = \ell(a) + 1$, which gives $\ell(a') = \ell(b') + 1$ and in particular $\ell(a') \geq 1$. It follows that at least one $a'_j \neq e$ i.e. $c_j = b_j \neq a_j$.

Recall also that $\alpha a = \beta b = \gamma$ from Proposition 6.5.31. Therefore a and b are in $EndMon_n(\alpha a)$ and it follows from Lemma 4.3.9 that $\Delta(a, b)$ is in $EndMon_n(\alpha a)$. From Lemma 6.5.23 $\Delta(a, b) = c$ so it follows that c is in $EndMon_n(\alpha a)$ i.e. we have for some α' in A_n^+ with $\ell(\alpha') < \ell(\alpha)$ that

$$\alpha a = \alpha'(c) = \alpha'(a'a).$$

By cancellation of a we have $\alpha = \alpha' a'$.

Consider the maximal j for which $c_j = b_j \neq a_j$. Then $a'_{j+1} = \cdots = a'_{p+1} = e$, i.e. $a' = a'_2 \ldots a'_j$. It follows that the facet $[\![\alpha a_j \sigma_{j-1} \ldots \sigma_2]\!]_1$ satisfies

$$\begin{aligned}
[\![(\alpha) a_j \sigma_{j-1} \ldots \sigma_2]\!]_1 &= [\![(\alpha' a') a_j \sigma_{j-1} \ldots \sigma_2]\!]_1 \\
&= [\![(\alpha' a'_2 \ldots a'_j) a_j \sigma_{j-1} \ldots \sigma_2]\!]_1 \\
&= [\![\alpha' a'_2 \ldots (a'_j a_j) \sigma_{j-1} \ldots \sigma_2]\!]_1 \\
&= [\![\alpha' a'_2 \ldots (c_j) \sigma_{j-1} \ldots \sigma_2]\!]_1 \\
&= [\![\alpha' a'_2 \ldots a'_{j-1} (b_j) \sigma_{j-1} \ldots \sigma_2]\!]_1.
\end{aligned}$$

Post multiplication by $b_j \sigma_{j-1} \ldots \sigma_2$ corresponds to the face map $\partial^{n-1}_{l_j+j-2}$ (recall $\ell(b_j) = l_j$). Therefore $[\![\alpha a_j \sigma_{j-1} \ldots \sigma_2]\!]_1$ is a facet of $[\![\alpha' a'_2 \ldots a'_{j-1}]\!]_0$ and we have that $\ell(\alpha' a'_2 \ldots a'_{j-1}) < \ell(\alpha)$ since $\alpha = \alpha' a'_2 \ldots a'_j$ and $\ell(a'_j) \geq 1$ ($c_j = b_j = a'_j a_j \neq a_j$). Therefore $[\![\alpha a_j \sigma_{j-1} \ldots \sigma_2]\!]_1$ is in $\mathcal{C}^n(k)$. ☐

6.5.33. Proof of Point A: Proof of case (iii): $\ell(\beta b) > \ell(\alpha a)$.

Proposition 6.5.34. *Recall that for some u and v in A_p^+, and γ in A_n^+ with $EndMon_p(\gamma) = \emptyset$, that $\alpha a = \gamma u$ and $\beta b = \gamma v$. If we are in case (iii) then $b \neq e$. Furthermore, we only need to consider the case when $\gamma = \alpha a$ so $\beta b = \gamma v = \alpha a v$. In this case it follows $EndGen_p(\beta b) \neq \emptyset$.*

Proof. Case (iii) states that $\ell(\beta b) > \ell(\alpha a)$, and note that this can only happen when $b \neq e$ since $\ell(\beta) < \ell(\alpha)$. Recall this implies $\ell(\beta) = \ell(\alpha) - 1$ from Lemma 6.5.16. If $u \neq e$ then αa satisfies $EndGen_p(\alpha a) \neq \emptyset$. By Remark 6.5.26 it was under this hypothesis that we proved case (i), i.e. if this holds then we have proved in the proof of case (i) that a facet containing $[\![\alpha a]\!]_p$ lies in $\mathcal{C}^n(k)$. Therefore we can assume $u = e$. Then $\alpha a = \gamma$ and it follows that $\beta b = \gamma v = \alpha a v$. Since $\ell(\beta b) > \ell(\alpha a)$ it follows $\ell(v) \geq 1$ and therefore $EndGen_p(\beta b) \neq \emptyset$. ☐

Proposition 6.5.35. *If $EndGen_0(\beta b) \neq \emptyset$, this contradicts the choice of b, i.e. we chose b such that $\sum_{k=2}^{p+1} l_k$ was minimal, as in Lemma 6.5.16.*

Proof. Let τ in $EndGen_0(\beta b)$. Then since τ letterwise commutes with $b_2 \ldots b_{p+1}$ it follows that τ is in $EndGen_0(\beta)$ from Lemma 4.3.12. Then $\beta = \beta' \tau$ for some β' in A_n^+ with $\ell(\beta') < \ell(\beta)$. Then

$$\begin{aligned}
[\![\beta b]\!]_p &= [\![(\beta' \tau) b]\!]_p \\
&= [\![\beta' \tau b]\!]_p \\
&= [\![\beta' b \tau]\!]_p \\
&= [\![\beta' b]\!]_p
\end{aligned}$$

which contradicts our choice of b, since β' can be enlarged to $\ell(\alpha) - 1$, by including the leftmost generator from b, and this would reduce the length of b. ☐

PROPOSITION 6.5.36. *If* σ_1 *is in* $EndGen_p(\beta b) \neq \emptyset$*, this contradicts the choice of* b*, i.e. we chose* b *such that* $\sum_{k=2}^{p+1} l_k$ *was minimal, as in Lemma 6.5.16.*

PROOF. If σ_1 is in $EndGen_p(\beta b)$, then since σ_1 letterwise commutes with $b_3 \dots b_{p+1}$ it follows that σ_1 is in $EndGen_p(\beta b_2)$ by Lemma 4.3.12. From Lemma 6.5.20, $\Delta(\sigma_1, b_2) = b_2\sigma_1 b_2$ and by Lemma 4.3.11 this is in $EndMon_n(\beta b_2)$, giving by cancellation of b_2 that $b_2\sigma_1$ is in $EndMon_n(\beta)$. So $\beta = \beta' b_2 \sigma_1$ for some β' in A_n^+. Then

$$\begin{aligned} [\![(\beta)(b)]\!]_p &= [\![(\beta' b_2\sigma_1)(b)]\!]_p \\ &= [\![(\beta' b_2\sigma_1)(b_2 \dots b_{p+1})]\!]_p \end{aligned}$$

and by Lemma 6.5.21, $b_2\sigma_1 b_2$ can be written as $\hat{b}_1\sigma_1 b_2\sigma_1$ where here we note that b_1 in the notation Lemma 6.5.21 acts as σ_1 here. So we have

$$\begin{aligned} [\![(\beta)(b)]\!]_p &= [\![(\beta' b_2\sigma_1)(b_2 \dots b_{p+1})]\!]_p \\ &= [\![\beta'(b_2\sigma_1 b_2)(b_3 \dots b_{p+1})]\!]_p \\ &= [\![\beta'(\hat{b}_1\sigma_1 b_2\sigma_1)(b_3 \dots b_{p+1}]\!]_p) \\ &= [\![\beta'(\hat{b}_1\sigma_1 b_2)(b_3 \dots b_{p+1})\sigma_1]\!]_p \\ &= [\![\beta'(\hat{b}_1\sigma_1)(b_2 b_3 \dots b_{p+1})\sigma_1]\!]_p \\ &= [\![\beta'\hat{b}_1\sigma_1(b)\sigma_1]\!]_p \\ &= [\![\beta'\hat{b}_1\sigma_1 b]\!]_p \end{aligned}$$

with $\ell(\beta'\hat{b}\sigma_1) < \ell(\beta)$. This contradicts the choice of b as in Proposition 6.5.35. □

PROPOSITION 6.5.37. *If* $EndGen_p(\beta b) \neq \emptyset$ *but* $EndGen_1(\beta b) = \emptyset$*, this contradicts the choice of* b*, i.e. we chose* b *such that* $\sum_{k=2}^{p+1} l_k$ *was minimal, as in Lemma 6.5.16.*

PROOF. If $EndGen_p(\beta b) \neq \emptyset$ but $EndGen_1(\beta b) = \emptyset$, then σ_j is in $EndGen_p(\beta b) = EndGen_p(\beta(b_2 \dots b_{p+1}))$ for some $2 \leq j \leq p$. Since σ_j letterwise commutes with $b_{j+2} \dots b_{p+1}$ it follows from Lemma 4.3.12 that σ_j is in $EndGen_p(\beta b_2 \dots b_{j+1})$. From Lemma 6.5.20, $\Delta(\sigma_j, b_{j+1}) = b_{j+1}\sigma_j b_{j+1}$ and by Lemma 4.3.11 this is in $EndMon_n(\beta b_2 \dots b_{j+1})$, giving by cancellation of the b_{j+1} that $b_{j+1}\sigma_j$ is in $EndMon_n(\beta b_2 \dots b_j)$. By Lemma 6.5.28, $b_{j+1}\sigma_j = \bar{b}_j b_j$ and so by cancellation of b_j, $\bar{b}_j$ is in $EndMon_n(\beta b_2 \dots b_{j-1})$. From Lemma 4.3.12, since $\bar{b}_j$ letterwise commutes with $b_2 \dots b_{j-1}$ we have $\bar{b}_j$ is in $EndMon_n(\beta)$ so $\beta = \beta'\bar{b}_j$ for some β'

in A_n^+ with $\ell(\beta') < \ell(\beta)$. Then it follows that

$$\begin{aligned}
[\![(\beta)b]\!]_p &= [\![(\beta'\bar{b}_j)(b)]\!]_p \\
&= [\![(\beta'\bar{b}_j)(b_2\dots b_{p+1})]\!]_p \\
&= [\![(\beta'\bar{b}_j)(b_2\dots b_{j-1})b_j(b_{j+1}\dots b_{p+1})]\!]_p \\
&= [\![(\beta')(b_2\dots b_{j-1})(\bar{b}_jb_j)(b_{j+1}\dots b_{p+1})]\!]_p \\
&= [\![\beta'(b_2\dots b_{j-1})(b_{j+1}\sigma_j)(b_{j+1}\dots b_{p+1})]\!]_p \text{ since } \bar{b}_jb_j = b_{j+1}\sigma_j \\
&= [\![\beta'(b_2\dots b_{j-1})(b_{j+1}\sigma_jb_{j+1})(b_{j+2}\dots b_{p+1})]\!]_p \\
&= [\![\beta'(b_2\dots b_{j-1})(\hat{b}_jb_jb_{j+1}\sigma_j)(b_{j+2}\dots b_{p+1})]\!]_p \quad \text{since } b_{j+1}\sigma_jb_{j+1} = \hat{b}_jb_jb_{j+1}\sigma_j \text{ by Lemma 6.5.21} \\
&= [\![\beta'(b_2\dots b_{j-1})(\hat{b}_j)(b_jb_{j+1})(\sigma_j)(b_{j+2}\dots b_{p+1})]\!]_p \\
&= [\![\beta'\hat{b}_j(b_2\dots b_{j-1})(b_jb_{j+1})\sigma_j(b_{j+2}\dots b_{p+1})]\!]_p \\
&= [\![\beta'\hat{b}_j(b_2\dots b_{j-1}b_jb_{j+1}b_{j+2}\dots b_{p+1})\sigma_j]\!]_p \\
&= [\![\beta'\hat{b}_j(b)\sigma_j]\!]_p \\
&= [\![\beta'\hat{b}_jb]\!]_p
\end{aligned}$$

with $\ell(\beta'\hat{b}_j) < \ell(\beta)$, since $\ell(\beta b) = \ell((\beta'\hat{b}_j)b\sigma_j)$, giving $\ell(\beta) = \ell((\beta'\hat{b}_j)\sigma_j)$. This contradicts the choice of b as in Proposition 6.5.35. ☐

This concludes the proof of case (iii) and hence the proof of Point A.

6.5.38. Proof of Point B. Recall Point B: If $\ell(\alpha) = \ell(\beta) = k+1$ then $[\![\alpha]\!]_0 \cap [\![\beta]\!]_0 \subseteq \mathcal{C}^n(k)$.

PROPOSITION 6.5.39. *Suppose $\alpha \neq \beta$ in A_n^+. If $\ell(\alpha) = \ell(\beta) = k+1$ then either $[\![\alpha]\!]_0 \cap [\![\beta]\!]_0 = \emptyset$ or $[\![\alpha]\!]_0 \cap [\![\beta]\!]_0 \subseteq \mathcal{C}^n(k)$.*

PROOF. Suppose $[\![\alpha]\!]_0 \cap [\![\beta]\!]_0 \neq \emptyset$. Then there exists a and b as defined in Definition 6.5.17 such that $[\![\alpha a]\!]_p = [\![\beta b]\!]_p$ for some $1 \leq p \leq n-1$. It follows that there exists some γ in A_n^+ and u, v in A_p^+ such that

$$\alpha a = \gamma u \text{ and } \beta b = \gamma v.$$

Suppose that $u \neq e$. Then by the proof of Point A case (i) it follows that a facet of $[\![\alpha]\!]_0$ containing $[\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$, as it was under this hypothesis that we proved case (i) (see Remark 6.5.26). Hence $[\![\alpha a]\!]_p = [\![\beta b]\!]_p$ itself is in $\mathcal{C}^n(k)$. Similarly if $v \neq e$ then a facet of $[\![\beta]\!]_0$ containing $[\![\beta b]\!]_p = [\![\alpha a]\!]_p$ is in $\mathcal{C}^n(k)$, and hence $[\![\beta b]\!]_p = [\![\alpha a]\!]_p$ itself is in $\mathcal{C}^n(k)$. So we are left with the case that $u = v = e$, giving

$$\alpha a = \gamma = \beta b$$

and since $\ell(\alpha) = \ell(\beta)$ it follows that $\ell(a) = \ell(b)$. Since $\alpha \neq \beta$ it follows $a \neq b$. Recall from Definition 6.5.22 there exists $c = c_2 \dots c_{p+1}$ and from Lemma 6.5.23 $c = \Delta(a, b)$ and

$c = a'a = b'b$. Since $\ell(a) = \ell(b)$ then $\ell(a') = \ell(b')$. Suppose $a' = e$, then $\ell(a') = \ell(b')$ gives $b' = e$ and hence $c = a = b$. But $a \neq b$ so it follows that $a' \neq e$ and in particular $\ell(a') \geq 1$.

From Lemma 4.3.11, since a and b are in $EndMon_n(\alpha a)$ it follows that $\Delta(a, b) = c$ is in $EndMon_n(\alpha a)$, so $\alpha a = \alpha' c = \alpha'(a'a)$ for some α' in A_n^+. By cancellation of a we have $\alpha = \alpha' a'$ and $\ell(\alpha') < \ell(\alpha)$. Then

$$\begin{aligned} [\![\alpha a]\!]_p &= [\![(\alpha' a')a]\!]_p \\ &= [\![\alpha' c]\!]_p \end{aligned}$$

and $[\![\alpha' c]\!]_p$ is in $\mathcal{C}^n(k)$ since c represents a series of face maps originating at $[\![\alpha']\!]_0$, with each face map given by the map corresponding to left multiplication by c_j, which is either the face map corresponding to a_j or b_j. $\square$

This completes the proof of B, and hence by Proposition 6.5.8 it follows that $\|\mathcal{C}^n_\bullet\|$ is $(n-2)$ connected.

6.6. Proof of Theorem C

6.6.1. Results on face and stabilisation maps. Recall the definition of the face maps of $\mathcal{A}^n_\bullet$ from Definition 6.4.4:

$$\partial_k^p : \mathcal{A}_p^n \to \mathcal{A}_{p-1}^n \text{ for } 0 \leq k \leq p$$

and given by

$$\begin{aligned} \partial_k^p : \mathcal{A}_p^n &\to \mathcal{A}_{p-1}^n \\ \partial_k^p : A_n^+ \bkslash\!\!\bkslash \mathcal{C}_p^n &\to A_n^+ \bkslash\!\!\bkslash \mathcal{C}_{p-1}^n \end{aligned}$$

where ∂_k^p is induced by the face maps of $\mathcal{C}^n_\bullet$, which are a composite of right multiplication of the representative for the equivalence class in $\mathcal{C}_p^n$ by $(\sigma_{n-p+k}\sigma_{n-p+k-1}\ldots\sigma_{n-p+1})$, before the inclusion to the equivalence class in $\mathcal{C}_{p-1}^n$.

Lemma 6.6.2. *The face maps ∂_k^p of $\mathcal{A}^n_\bullet$ are all homotopic to the zeroth face map ∂_0^p.*

Proof. Recall from Lemma 6.4.3 that for each $0 \leq p \leq n-1$ there is a homotopy equivalence

$$A_n^+ /\!\!/ A_{n-p-1}^+ \simeq A^+(n; n-p-1) = \mathcal{C}_p^n,$$

with the map defined levelwise on the bar construction by

$$\begin{aligned} B_k(A_n^+, A_{n-p-1}^+, *) &\to A^+(n; n-p-1) \\ \alpha[m_1, \ldots, m_k] &\mapsto \overline{\alpha} \end{aligned}$$

where $\alpha \in A_n^+$, $m_i \in A_{n-p-1}^+$ for all i and $\alpha = \overline{\alpha}\beta$ for $\overline{\alpha} \in A^+(n; n-p-1)$ and $\beta \in A_{n-p-1}^+$. Define the map

$$d_k^p : A_n^+ \bkslash\!\!\bkslash A_n^+ /\!\!/ A_{n-p-1}^+ \to A_n^+ \bkslash\!\!\bkslash A_n^+ /\!\!/ A_{n-p}^+$$

to be the composition of two maps $\iota_p \circ \bar{d}_k^p$. The first map

$$\bar{d}_k^p : A_n^+ \mathbin{/\!\!/\!} A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+ \to A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+$$

is given by right multiplication of the central term in the double homotopy quotient by $(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})$. The set of (j,k)-simplices in $A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+$ is given by $(A_n^+)^j \times A_n^+ \times (A_{n-p-1}^+)^k$ and an element in this set is given by $[a_1, \dots, a_j]a[a_1', \dots, a_k']$ where a_i and a are in A_n^+ and a_i' are in A_{n-p-1}^+. The map $\bar{d}_k^p$ acts on this simplex as

$$\bar{d}_k^p([a_1, \dots, a_j]a[a_1', \dots, a_k']) = [a_1, \dots, a_j]a(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})[a_1', \dots, a_k']$$

and since $(\sigma_{n-p+k}\sigma_{n-p+k-1}\dots\sigma_{n-p+1})$ letterwise commutes with all words in A_{n-p-1}^+, it follows that $\bar{d}_k^p$ commutes with all face maps of the bi-semi-simplicial set $A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+$. Therefore the map on the central term of each simplex gives a map on the whole bi-semi-simplicial set, and hence its geometric realisation: the double homotopy quotient $A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+$. The second map ι_p is given by the map

$$\iota_p : A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+ \to A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p}^+$$

induced by the inclusion $A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$. Note here that d_0^p satisfies $\bar{d}_0^p$ is the identity map, and therefore $d_0^p = \iota_p$. Then the diagram

$$\begin{tikzcd}
A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+ \arrow[r, "\simeq"] \arrow[d, "d_k^p"'] & A_n^+ \backslash\!\!\backslash \mathcal{C}_p^n \arrow[d, "\partial_k^p"] \\
A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p}^+ \arrow[r, "\simeq"] & A_n^+ \backslash\!\!\backslash \mathcal{C}_{p-1}^n
\end{tikzcd}$$

commutes for all $p \geq 0$. The map $\bar{d}_k^p$ restricted to $A_n^+ \backslash\!\!\backslash A_n^+$ is A_{n-p-1}^+-equivariant, and so is the identity map $id_{A_n^+ \backslash\!\!\backslash A_n^+}$. Applying Proposition 4.5.21 to these two maps therefore gives an A_{n-p-1}^+-equivariant homotopy between them. It follows that they induce homotopic maps $\bar{d}_k^p$ and $id_{A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+}$ on $A_n^+ \backslash\!\!\backslash A_n^+ \mathbin{/\!\!/} A_{n-p-1}^+$. Applying the inclusion ι_p, to both maps and the homotopy gives a homotopy h_k from d_k^p to ι_p. However ι_p is precisely the map d_0^p, and thus h_k is a homotopy from d_k^p to d_0^p for all k. Then the image of h_k under the homotopy equivalence yields a homotopy from ∂_k^p to the zeroth face map ∂_0^k, as required. ☐

LEMMA 6.6.3. *Under the homotopy equivalence $\mathcal{A}_p^n \simeq BA_{n-p-1}^+$ of Lemma 6.4.6, the zeroth face map $\partial_0^p : \mathcal{A}_p^n \to \mathcal{A}_{p-1}^n$ is mapped to the map $s_* : BA_{n-p-1}^+ \to BA_{n-p}^+$ induced by the stabilisation map $s : A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$.*

PROOF. From Lemma 6.4.6, Lemma 6.4.3 and the proof of the previous Lemma 6.6.2 we have the following

$$
\begin{tikzcd}
BA_{n-p-1} \arrow[d, "s_*"] & A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p-1}^+ \arrow[l, "\simeq", "f^p"'] \arrow[r, "\simeq"] \arrow[d, "d_k^p"'] & A_n^+ \mathbin{\backslash\!\backslash} \mathcal{C}_p^n \arrow[d, "\partial_k^p"] \\
BA_{n-p} & A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p}^+ \arrow[l, "\simeq", "f^{p-1}"'] \arrow[r, "\simeq"] & A_n^+ \mathbin{\backslash\!\backslash} \mathcal{C}_{p-1}^n
\end{tikzcd}
$$

where the map from the centre to the left is given on the (j,k)-simplices of the geometric realisation by

$$
\begin{array}{rcl}
f_{(j,k)}^p : (A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p-1}^+)_{(j,k)} & \to & (* \mathbin{/\!/} A_{n-p-1}^+)_k \\
{[a_1,\ldots,a_j]a[a'_1,\ldots,a'_k]} & \mapsto & *[a'_1,\ldots,a'_k]
\end{array}
$$

where a and a_i are in A_n^+ and a'_i is in A_{n-p-1}^+. The map d_0^p is the map

$$d_0^p : A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p-1}^+ \to A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p}^+$$

induced by the inclusion $A_{n-p-1}^+ \hookrightarrow A_{n-p}^+$. Restricting this map to (j,k)-simplices of the double homotopy quotient gives

$$
\begin{array}{rcl}
(d_0^p)_{(j,k)} : (A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p-1}^+)_{(j,k)} & \to & (A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p}^+)_{(j,k)} \\
{[a_1,\ldots,a_j]a[a'_1,\ldots,a'_k]} & \mapsto & [a_1,\ldots,a_j]a[a'_1,\ldots,a'_k]
\end{array}
$$

where a and a_i are in A_n^+ and a'_i is in A_{n-p-1}^+, hence a'_i is in A_{n-p}^+. Applying this map before the homotopy equivalence to the classifying space gives

$$
\begin{tikzcd}
(A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p-1}^+)_{(j,k)} \arrow[r, "(d_0^p)_{(j,k)}"] \arrow[d, "f_{(j,k)}^p"'] & (A_n^+ \mathbin{\backslash\!\backslash} A_n^+ \mathbin{/\!/} A_{n-p}^+)_{(j,k)} \arrow[d, "f_{(j,k)}^{p-1}"] \\
(* \mathbin{/\!/} A_{n-p-1}^+)_k \arrow[d, equal] & (* \mathbin{/\!/} A_{n-p}^+)_k \arrow[d, equal] \\
BA_{n-p-1}^+ \arrow[r, dotted] & BA_{n-p}^+
\end{tikzcd}
$$

and on a (j,k) simplex this map is given by

$$
\begin{tikzcd}
{[a_1,\ldots,a_j]a[a'_1,\ldots,a'_k]} \arrow[r, mapsto, "(d_0^p)_{(j,k)}"] \arrow[d, mapsto, "f_{(j,k)}^p"'] & {[a_1,\ldots,a_j]a[a'_1,\ldots,a'_k]} \arrow[d, mapsto, "f_{(j,k)}^{p-1}"] \\
*[a'_1,\ldots,a'_k] \arrow[r, mapsto, dotted] & *[a'_1,\ldots,a'_k].
\end{tikzcd}
$$

We note that the dotted map is precisely the map which defines the natural inclusion $BA^+_{n-p-1} \to BA^+_{n-p}$ under the identification of $* /\!\!/ A^+_r$ with BA^+_r for all r. The natural inclusion is in turn induced by the stabilisation map $A^+_r \overset{s}{\hookrightarrow} A^+_{r+1}$ and so we denote it s_*. Therefore under the homotopy equivalence between the classifying space and the double homotopy quotient, d^p_0 is equivalent to s_*. From the proof of the previous lemma, under the homotopy equivalence between the double homotopy quotient and $\mathcal{C}^n_p$ for each p, d^p_0 is the map induced by ∂^p_0 and therefore it follows that ∂^p_0 is equivalent to s_* under the homotopy equivalences $C^n_p \simeq BA^+_{n-p-1}$ for each p. $\square$

6.6.4. Spectral sequence argument. In this section we run first a quadrant spectral sequence for filtration of $\|\mathcal{A}^n_\bullet\|$, as in [**41**, 2 (sSS)]. Recall the four points we proved regarding $\|\mathcal{A}^n_\bullet\|$:

(1) $\mathcal{A}^n_\bullet$ is built out of spaces $\mathcal{A}^n_p$ for $p \geq 0$
(2) there exist homotopy equivalences $\mathcal{A}^n_p \simeq BA^+_{n-p-1}$ for $p \geq 0$
(3) there is a map from the geometric realisation of $\mathcal{A}^n_\bullet$ to the classifying space BA^+_n, which we call $\|\phi_\bullet\|$
$$\|\mathcal{A}^n_\bullet\| \overset{\|\phi_\bullet\|}{\to} BA^+_n$$
(4) $\|\phi_\bullet\|$ is $(n-1)$ connected, i.e. it is an isomorphism on homotopy groups π_r for $0 \leq r \leq (n-2)$, and a surjection for $r = (n-1)$.

The first quadrant spectral sequence of the filtration of $\|\mathcal{A}^n_\bullet\|$ satisfies
$$E^1_{k,l} = H_l(\mathcal{A}^n_k) \Rightarrow H_{k+l}(\|\mathcal{A}^n_\bullet\|).$$
By point (2) the left hand side is given by $E^1_{k,l} = H_l(\mathcal{A}^n_k) = H_l(BA^+_{n-k-1})$. The first page of the spectral sequence is therefore as in Figure 3. By points (3) and (4) the highly connected map $\|\phi_\bullet\|$ gives that the right hand side satisfies
$$\begin{aligned} H_{k+l}(\|\mathcal{A}^n_\bullet\|) &\cong H_{k+l}(BA^+_n) \quad &\text{when } (k+l) < n-1 \\ H_{k+l}(\|\mathcal{A}^n_\bullet\|) &\twoheadrightarrow H_{k+l}(BA^+_n) \quad &\text{when } (k+l) = n-1. \end{aligned}$$
The differential d^1 is given by an alternating sum of face maps in $\mathcal{A}^n_\bullet$. By Corollary 6.6.2 the face maps are all homotopic to each other and by Lemma 6.6.3 they are all homotopic to the stabilisation map s_*, via the homotopy equivalence $\mathcal{A}^n_p \simeq BA^+_{n-p-1}$. Therefore the alternating sum of face maps in the differential d^1 will cancel out to give the zero map when there are an even number of terms, and will give the stabilisation map when there are an odd number of terms, i.e.
$$\begin{aligned} d^1 : E^1_{\text{even},l} \to E^1_{\text{odd},l} \quad & \text{odd number of terms, so equals the stabilisation map } s \\ d^1 : E^1_{\text{odd},l} \to E^1_{\text{even},l} \quad & \text{even number of terms, so equals the zero map } 0 \end{aligned}$$
which gives the E^1 page as shown in Figure 4.

$$
\begin{array}{c|ccccccc}
\vdots & \vdots & & \vdots & & \vdots & & \\
3 & H_3(BA^+_{n-1}) & \xleftarrow{d^1} & H_3(BA^+_{n-2}) & \xleftarrow{d^1} & H_3(BA^+_{n-3}) & & \cdots \\
2 & H_2(BA^+_{n-1}) & \xleftarrow{d^1} & H_2(BA^+_{n-2}) & \xleftarrow{d^1} & H_2(BA^+_{n-3}) & & \cdots \\
1 & H_1(BA^+_{n-1}) & \xleftarrow{d^1} & H_1(BA^+_{n-2}) & \xleftarrow{d^1} & H_1(BA^+_{n-3}) & & \cdots \\
0 & H_0(BA^+_{n-1}) & \xleftarrow{d^1} & H_0(BA^+_{n-2}) & \xleftarrow{d^1} & H_0(BA^+_{n-3}) & & \cdots \\
\hline
 & 0 & & 1 & & 2 & &
\end{array}
$$

FIGURE 3. The E^1 page of the spectral sequence.

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & \vdots & & \vdots & \\
3 & H_3(BA^+_{n-1}) & \xleftarrow{0} & H_3(BA^+_{n-2}) & \xleftarrow{s_*} & H_3(BA^+_{n-3}) & \xleftarrow{0} \cdots \\
2 & H_2(BA^+_{n-1}) & \xleftarrow{0} & H_2(BA^+_{n-2}) & \xleftarrow{s_*} & H_2(BA^+_{n-3}) & \xleftarrow{0} \cdots \\
1 & H_1(BA^+_{n-1}) & \xleftarrow{0} & H_1(BA^+_{n-2}) & \xleftarrow{s_*} & H_1(BA^+_{n-3}) & \xleftarrow{0} \cdots \\
0 & H_0(BA^+_{n-1}) & \xleftarrow{0} & H_0(BA^+_{n-2}) & \xleftarrow{s_*} & H_0(BA^+_{n-3}) & \xleftarrow{0} \cdots \\
\hline
 & 0 & & 1 & & 2 &
\end{array}
$$

FIGURE 4. The E^1 page of the spectral sequence, with differentials filled in.

We proceed by induction, assuming that homological stability holds for previous groups in the sequence, i.e. the map induced on homology by the stabilisation map s_*

$$H_i(BA^+_{k-1}) \to H_i(BA^+_k)$$

is an isomorphism for $k > 2i$ and is a surjection for $k = 2i$ whenever $k < n$.

Here we note that the result holds for the base case $n = 1$, since we have to check $H_0(BA_0^+) \to H_0(BA_1^+)$ is a surjection, which is true since BA_n^+ is connected for all n, and so in fact $H_0(BA_0^+) \to H_0(BA_1^+)$ is an isomorphism.

LEMMA 6.6.5. *Under the inductive hypothesis, the spectral sequence satisfies that the $E_{0,l}$ terms stabilise on the E^1 page for $2l \leq n$, i.e.*

$$E^1_{0,l} = E^\infty_{0,l} \text{ when } 2l \leq n.$$

In particular the d^1 differential does not alter these groups, and all possible sources of differentials mapping to $E_{0,l}$ for $2l \leq n$ are trivial from the E^2 page.

PROOF. The d^1 differentials are given by either the zero map or the stabilisation map as shown in Figure 4. The d^1 differentials

$$d^1 : E^1_{0,l} \to E^1_{1,l}$$

are given by the zero map, and the $E^1_{-1,l}$ terms are zero, due to the fact that this is a first quadrant spectral sequence. This gives that the $E^2_{0,l}$ terms are equal to the $E^1_{0,l}$ terms.

To show that the sources of all other differentials to $E_{0,l}$ for $2l \leq n$ are zero, we invoke the inductive hypothesis. This gives that the stabilisation maps, or d^1 differentials going from even to odd columns are isomorphisms on the E^1 page, in the interior of the triangle of height $\lfloor \frac{n}{2} \rfloor$ and base n, and surjections on the diagonal. Since the d^1 differentials going from the odd to the even columns are zero, it follows that all terms in this triangle are zero on the E^2 page, except the ones on the zero column. These groups are precisely the sources of differentials to $E_{0,l}$ for $2l \leq n$. □

We are now in a position to prove the desired result.

THEOREM 6.6.6. *The sequence of monoids A_n^+ satisfies homological stability, that is*

$$H_i(BA^+_{n-1}) \cong H_i(BA^+_n)$$

when $2i < n$, and the map $H_i(BA^+_{n-1}) \to H_i(BA^+_n)$ is surjective when $2i = n$.

PROOF. From Lemma 6.6.5, the spectral sequence satisfies

$$E^\infty_{0,i} = E^1_{0,i} = H_i(BA^+_{n-1})$$

when $2i \leq n$. From Proposition 6.4.9 and Theorem 6.5.1

$$H_i(\|\mathcal{A}^n_\bullet\|) \cong H_i(BA^+_n)$$

when $i \leq n-2$, and the map $H_i(\|\mathcal{A}^n_\bullet\|) \to H_i(BA^+_n)$ is onto when $i = n-1$. The spectral sequence abuts to $H_{k+l}(\|\mathcal{A}^n_\bullet\|)$ and from Lemma 6.6.5 the only non zero groups on the diagonal $E^\infty_{k,l}$ when $k + l = i$ are the groups $E^\infty_{0,i}$. Putting these results together we get

$$H_i(BA^+_{n-1}) = E^\infty_{0,i} = H_{i+0}(\|\mathcal{A}^n_\bullet\|) \cong H_i(BA^+_n)$$

when both $i \leq \frac{n}{2}$ and $i \leq n-2$ are satisfied. When $n \geq 2$, $i < \frac{n}{2}$ implies $i \leq n-2$ and the case $n = 1$ was handled as the base case of the inductive hypothesis. Therefore we have that an isomorphism is induced when $2i < n$.

When $i \leq n-1$ and $i \leq \frac{n}{2}$ we have

$$H_i(BA^+_{n-1}) = E^\infty_{0,i} = H_{i+0}(\|\mathcal{A}^n_\bullet\|) \twoheadrightarrow H_i(BA^+_n)$$

and for $n \geq 2$, $i < \frac{n}{2}$ implies $i \leq n-1$. Again the case $n = 1$ was handled as the base case of the inductive hypothesis. This gives the required range for the surjection, and hence completes the proof. $\square$

APPENDIX A

Python calculations

A.1. Code

Below is pseudo-code for the Python code used in Chapter 2, which uses the program PyCox by Geck [**26**], and requires the PyCox Python file *chv.py*. It is followed by some example calculations, which are referred to in the text. Many thanks to Edmund Howse, who showed me how to use PyCox and provided example code and computations for me to work from. The code file can be found on my (current) web-page.

- **cosetreps(**W, I**)**: Given a Coxeter group W and a subset of its simple reflections I returns a list of all distinguished right coset representatives of W_I in W.
- **leftcosetreps(**W, I**)**: Given a Coxeter group W and a subset of its simple reflections I returns a list of all distinguished left coset representatives of W_I in W.
- **cosetlengths(**W, I**)**: Given a Coxeter group W and a subset of its simple reflections I, returns the length of the distinguished right coset representatives of W_I in I as a list.
- **leftDS(**W, X**)**: Given a Coxeter group W and X a set of words in W, returns the left descent set in W for each word in X, in a list.
- **intersect(**a, b**)**: Returns the intersection of two lists a and b.
- **collapse(**W, I, w**)**: Given a Coxeter group W, a subset of its simple reflections I and a simple reflection w in W, computes the following:
  - X: distinguished right coset representatives of I in W
  - R: reduced words representing right multiplication of the words in list X by w
  - Y: the left decent set (generators the word can start in) of the words in list R
  - Z: the intersection of each entry of the list Y with I
  - L: the length of the coset representatives in list X
  - S: the length modulo 2 of the coset representatives in list X
  - A: a pair for each non-empty entry of Z, containing the entry of Z and the corresponding entry of S.

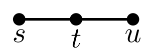
  Returns A, the data for the transfer and collapse map on generator corresponding to w, for subgroup corresponding to I.

- **paritycosetreps(W, I)**: Given a Coxeter group W and a subset of the simple reflections I, returns the number of distinguished coset representatives which have even length and the number of distinguished coset representatives which have odd length.
- **conjugateandlengths(W, I, w)**: Given a Coxeter group W, a subset of its simple reflections I and a simple reflection w, computes the conjugate of w by all distinguished coset representatives of W_I in W. Returns the conjugates which reduce to a simple generator of W, and the corresponding length modulo 2 of the conjugator.

Examples

This section consists of examples for all cases in the thesis for which we use the above Python code.

Example A.1 (For proof of Proposition 2.5.29 and Lemma 2.5.44). This example shows the code for the transfer and collapse map being used when W_T is $W(A_3)$:



and we consider the transfer from $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_3 = \langle\alpha\rangle$ which has generator $\alpha = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)$.
When $I = \{s, t\}$, our input to the Python module and the corresponding output is

```
>>> W = coxeter("A",3)
>>> collapse(W,[0,1],0)
[([0], 0), ([0], 1)]
>>> collapse(W,[0,1],1)
[([1], 0), ([0], 1)].
```

The first line of input sets the Coxeter group to be the inbuilt group $W(A_3)$ where generators s, t, u in the diagram are labelled $0, 1, 2$ respectively. The second line of input computes the transfer and collapse map of $(1 \otimes \Gamma_s)$, specified by the 0 in the third entry (corresponding to s). This is with respect to the subgroup generated by 0 and 1 (s and t) in the full group W. The output, $[([0], 0), ([0], 1)]$, is a list of pairs, the first entry in each pair corresponds to a generator and the second entry to its sign: 1 for negative and 0 for positive. So $([0], 0)$ corresponds to $+(1 \otimes \Gamma_s)$ and $([0], 1)$ to $-(1 \otimes \Gamma_s)$. The third line of input computes the transfer and collapse map in the same way for $(1 \otimes \Gamma_t)$, hence the 1 in the third input entry.

Putting these together we get:

$$
\begin{aligned}
\alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t\Big) \\
&\overset{f^1}{\mapsto} (1 \otimes \Gamma_s - 1 \otimes \Gamma_s) - (1 \otimes \Gamma_t - 1 \otimes \Gamma_s) \\
&= 1 \otimes \Gamma_s - 1 \otimes \Gamma_t
\end{aligned}
$$

as given in the proof of Proposition 2.5.29.

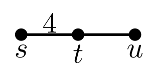
EXAMPLE A.2 (For proof of Lemma 2.5.55). We recall the formula for $\delta_k(e(\Gamma))$ from Equation (4).

$$
\delta_k(e(\Gamma)) = \sum_{\substack{i \geq 1 \\ |\Gamma_i| > |\Gamma_{i+1}|}} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subset \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)} \beta e(\Gamma')
$$

For the groups $W(B_3)$ and $W(H_3)$ in Lemma 2.5.55 the $\delta_4(\Gamma_{s,t,u \supset s})$ computation is given by:

$$
\delta_4(\Gamma_{s,t,u \supset s}) = \sum_{i=1,2} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subset \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)} \beta e(\Gamma').
$$

We use our 'conjugateandlengths' function to compute the distinguished coset representatives of a 2-generator subgroup W_I for $I \subset \{s, t, u\}$ and the conjugates of an element of $\{s, t, u\}$ by these representatives. If this conjugate is in $\{s, t, u\}$, the length modulo 2 of the conjugator is recorded. For instance when the group is $W(B_3)$



and the 2-generator subgroup is generated by $I = \{s, t\}$ with the element of $\{s, t, u\}$ being s, we input the following code:

```
>>> W = coxeter("B",3)
>>> conjugateandlengths(W,[0,1],0)
[([0], 0), ([0], 1), ([0], 0), ([0], 1)].
```

The output tells us that four coset representatives for W_I in W conjugate s to a generator of B_3. The first entry in each pair tells us this generator, and the second entry tells us the length modulo 2 of the corresponding coset representative. This corresponds to the sign of the coefficient, since it relies on the length (the sign is $+1$ if even length and -1 if odd length). In our example we see that four coset representatives conjugate s to itself, but there are two with even length and two with odd length. Therefore upon tensoring with $\mathbb{Z}$ over $W(B_3)$ in the proof of Lemma 2.5.55 the terms relating to these coset representatives will cancel out.

APPENDIX B

Calculations for Section 2.5

This Appendix contains proofs and calculations used in Section 2.5. The majority of these calculations compute twisted homology of finite Coxeter groups, using the De Concini - Salvetti resolution.

Proof of Example 2.5.5. Here differentials for flags containing only one generator are computed as in Example 2.5.3 and the other differentials are computed as follows.
We recall the formula for $\delta_k(e(\Gamma))$ from Equation (4). The differential $\delta_2(\Gamma_{s,t})$ is given by:

$$\begin{aligned}
\delta_2(\Gamma_{s,t}) &= \sum_{i=1}\sum_{\tau=s,t}\sum_{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}}(-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma') \\
&= \sum_{\beta\in W_{s,t}^{t}}(-1)^{\alpha(\Gamma,1,s,\beta)}\beta\Gamma_t+\sum_{\beta\in W_{s,t}^{s}}(-1)^{\alpha(\Gamma,1,t,\beta)}\beta\Gamma_s \\
&= \sum_{j=0}^{m(s,t)-1}(-1)^{\alpha(\Gamma,1,s,p(s,t;j))}p(s,t;j)\Gamma_t+\sum_{g=0}^{m(s,t)-1}(-1)^{\alpha(\Gamma,1,t,p(t,s;g))}p(t,s;g)\Gamma_s \\
&= \sum_{j=0}^{m(s,t)-1}(-1)^{j+1}p(s,t;j)\Gamma_t+\sum_{g=0}^{m(s,t)-1}(-1)^{g+2}p(t,s;g)\Gamma_s
\end{aligned}$$

where we recall that we define $p(s,t;j)$ to be the alternating product of s and t of length j, *ending* in an s (as opposed to $\pi(s,t;j)$ which is the alternating product *starting* in an s) e.g. $p(s,t;3)=sts$, $p(s,t;4)=tsts$, and compute $\alpha(\Gamma,1,\tau,\beta)$ as follows:

$$\begin{aligned}
\alpha(\Gamma_{s,t},1,s,p(s,t;j)) &= 1\ell(p(s,t;j))+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},s) \\
&= j+0+1 \\
&= j+1 \\
\alpha(\Gamma_{s,t},1,t,p(t,s;g)) &= 1\ell(p(t,s;g))+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},s) \\
&= g+0+2 \\
&= g+2.
\end{aligned}$$

The differential $\delta_3(\Gamma_{s,t\supset s})$ is given by:

$$
\begin{aligned}
\delta_3(\Gamma_{s,t\supset s}) &= \sum_{i=1,2}\sum_{\tau\in\Gamma_i}\sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma') \\
&= \sum_{\tau\in\{s,t\}}\sum_{\substack{\beta\in W_{\{s,t\}}^{\{s,t\}\setminus\{\tau\}}\\ \beta^{-1}s\beta\subset\{s,t\}\setminus\{\tau\}}} (-1)^{\alpha(\Gamma,1,\tau,\beta)}\beta e(\Gamma') + \sum_{\beta=e,s}(-1)^{\alpha(\Gamma,2,s,\beta)}\beta e(\Gamma') \\
&= \begin{cases} \sum_{\beta=e,p(t,s;m(s,t)-1)}((-1)^{\alpha(\Gamma,1,t,\beta)}\beta\Gamma_{s\supset s})+(-1)^3\Gamma_{st}+(-1)^5 s\Gamma_{st} & m(s,t)\text{ even} \\ \sum_{\beta=e}((-1)^{\alpha(\Gamma,1,t,\beta)}\beta\Gamma_{s\supset s})+\sum_{\beta=p(s,t;m(s,t)-1)}((-1)^{\alpha(\Gamma,1,s,\beta)}\beta\Gamma_{t\supset t}) \\ \quad +(-1)^3\Gamma_{st}+(-1)^5 s\Gamma_{st} & m(s,t)\text{ odd} \end{cases} \\
&= \begin{cases} (-1)^2\Gamma_{s\supset s}+(-1)^{m(s,t)+1}p(t,s;m(s,t)-1)\Gamma_{s\supset s}+(-1)^3\Gamma_{st}+(-1)^5 s\Gamma_{st} & m(s,t)\text{ even} \\ (-1)^2\Gamma_{s\supset s}+(-1)^{m(s,t)}p(s,t;m(s,t)-1)\Gamma_{t\supset t}+(-1)^3\Gamma_{st}+(-1)^5 s\Gamma_{st} & m(s,t)\text{ odd} \end{cases} \\
&= \begin{cases} \Gamma_{s\supset s}-p(t,s;m(s,t)-1)\Gamma_{s\supset s}-\Gamma_{st}-s\Gamma_{st} & m(s,t)\text{ even} \\ \Gamma_{s\supset s}-p(s,t;m(s,t)-1)\Gamma_{t\supset t}-\Gamma_{st}-s\Gamma_{st} & m(s,t)\text{ odd} \end{cases}
\end{aligned}
$$

and we compute $\alpha(\Gamma_{s,t\supset s},i,\tau,\beta)$ as follows

$$
\begin{aligned}
\alpha(\Gamma_{s,t\supset s},1,t,e) &= 1\cdot 0+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},t) \\
&= 0+0+2 \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\Gamma_{s,t\supset s},1,t,p(t,s;m(s,t)-1)) &= 1\cdot(m(s,t)-1)+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},t) \\
&= (m(s,t)-1)+0+2 \\
&= (m(s,t)-1)+2
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\Gamma_{s,t\supset s},1,s,p(s,t;m(s,t)-1)) &= 1\cdot(m(s,t)-1)+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},s) \\
&= (m(s,t)-1)+0+1 \\
&= m(s,t)
\end{aligned}
$$

$$\begin{aligned}\alpha(\Gamma_{s,t\supset s},2,s,e) &= 2\ell(e)+\sum_{k=1}^{1}|\Gamma_k|+\mu(\{s\},s)\\ &= 0+2+1\\ &= 3\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s,t\supset s},2,s,s) &= 2\ell(s)+\sum_{k=1}^{1}|\Gamma_k|+\mu(\{s\},s)\\ &= 2+2+1\\ &= 5.\end{aligned}$$

Similarly the differential for $\delta_3(\Gamma_{s,t\supset t})$ is given by:

$$\begin{aligned}\delta_3(\Gamma_{s,t\supset t}) &= \sum_{i=1,2}\sum_{\tau\in\Gamma_i}\sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}}(-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma')\\ &= \sum_{\tau\in\{s,t\}}\sum_{\substack{\beta\in W_{\{s,t\}}^{\{s,t\}\setminus\{\tau\}}\\ \beta^{-1}t\beta\subset\{s,t\}\setminus\{\tau\}}}(-1)^{\alpha(\Gamma,1,\tau,\beta)}\beta e(\Gamma')+\sum_{\beta=e,t}(-1)^{\alpha(\Gamma,2,t,\beta)}\beta e(\Gamma')\\ &= \begin{cases}(-1)^1\Gamma_{t\supset t}+(-1)^{m(s,t)}p(s,t;m(s,t)-1)\Gamma_{t\supset t}+(-1)^3\Gamma_{st}+(-1)^5t\Gamma_{st} & m(s,t)\text{ even}\\ (-1)^1\Gamma_{t\supset t}+(-1)^{m(s,t)+1}p(t,s;m(s,t)-1)\Gamma_{s\supset s}+(-1)^3\Gamma_{st}+(-1)^5t\Gamma_{st} & m(s,t)\text{ odd}\end{cases}\\ &= \begin{cases}(-1+p(s,t;m(s,t)-1))\Gamma_{t\supset t}-(1+t)\Gamma_{st} & m(s,t)\text{ even}\\ -\Gamma_{t\supset t}+p(t,s;m(s,t)-1)\Gamma_{s\supset s}-(1+t)\Gamma_{st} & m(s,t)\text{ odd}\end{cases}\end{aligned}$$

and we compute $\alpha(\Gamma_{s,t\supset s},i,\tau,\beta)$ as follows

$$\begin{aligned}\alpha(\Gamma_{s,t\supset t},1,s,e) &= 1\cdot 0+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},s)\\ &= 0+0+1+0\\ &= 1\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s,t\supset t},1,s,p(s,t;m(s,t)-1)) &= 1\cdot(m(s,t)-1)+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},s)\\ &= (m(s,t)-1)+0+1\\ &= m(s,t)\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s,t\supset t},1,t,p(t,s;m(s,t)-1)) &= 1\cdot(m(s,t)-1)+\sum_{k=1}^{0}|\Gamma_k|+\mu(\{s,t\},t)\\ &= (m(s,t)-1)+0+2\\ &= m(s,t)+1\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s,t\supset t},2,t,e) &= 2\ell(e)+\sum_{k=1}^{1}|\Gamma_k|+\mu(\{t\},t)\\ &= 0+2+1\\ &= 3\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma,2,t,t) &= 2\ell(t)+\sum_{k=1}^{1}|\Gamma_k|+\mu(\{t\},t)\\ &= 2+2+1\\ &= 5.\end{aligned}$$

□

Proof of Lemma 2.5.19. We compute using the De Concini resolution. From Example 2.5.6 we have:

$$\mathbb{Z}\underset{W_s}{\otimes}C_3 \xrightarrow{\delta_3} \mathbb{Z}\underset{W_s}{\otimes}C_2 \xrightarrow{\delta_2} \mathbb{Z}\underset{W_s}{\otimes}C_1 \xrightarrow{\delta_1} \mathbb{Z}\underset{W_s}{\otimes}C_0$$

Generators:

$1\otimes\Gamma_{s\supset s\supset s}$ $\quad 1\otimes\Gamma_{s\supset s}$ $\quad 1\otimes\Gamma_{s}$ $\quad 1\otimes\Gamma_{\emptyset}$

Differentials: $\quad 1\otimes\Gamma_s \longmapsto -2(1\otimes\Gamma_\emptyset)$

$$1\otimes\Gamma_{s\supset s}\longmapsto 0$$

$$1\otimes\Gamma_{s\supset s\supset s}\longmapsto 1\otimes -2(1\otimes\Gamma_{s\supset s})$$

Computing $H_2(W_t;\mathbb{Z}_t)=\frac{\ker(\delta_2)}{\mathrm{im}(\delta_3)}$ gives $\mathbb{Z}_2$, generated by $1\otimes\Gamma_{s\supset s}$. □

Proof of Lemma 2.5.20. We compute using the De Concini-Salvetti resolution. From Example 2.5.7 we have:

$$\mathbb{Z} \underset{W_T}{\otimes} C_3 \xrightarrow{\delta_3} \mathbb{Z} \underset{W_T}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_T}{\otimes} C_1$$

Generators:

$$\begin{array}{lll} 1 \otimes \Gamma_{s \supset s \supset s} & 1 \otimes \Gamma_{s \supset s} & 1 \otimes \Gamma_s \\ 1 \otimes \Gamma_{t \supset t \supset t} & 1 \otimes \Gamma_{t \supset t} & 1 \otimes \Gamma_t \\ 1 \otimes \Gamma_{s,t \supset s} & 1 \otimes \Gamma_{st} & \\ 1 \otimes \Gamma_{s,t \supset t} & & \end{array}$$

Differentials:

$$1 \otimes \Gamma_{s \supset s} \longmapsto 0$$
$$1 \otimes \Gamma_{t \supset t} \longmapsto 0$$
$$1 \otimes \Gamma_{s,t} \longmapsto m(s,t)(-(1 \otimes \Gamma_t) + (1 \otimes \Gamma_s))$$
$$1 \otimes \Gamma_{s \supset s \supset s} \longmapsto -2(1 \otimes \Gamma_{s \supset s})$$
$$1 \otimes \Gamma_{t \supset t \supset t} \longmapsto -2(1 \otimes \Gamma_{t \supset t})$$
$$1 \otimes \Gamma_{s,t \supset s} \longmapsto \begin{array}{ll} 2(1 \otimes \Gamma_{s \supset s}) & \text{if } m(s,t) \text{ even} \\ 1 \otimes \Gamma_{s \supset s} - 1 \otimes \Gamma_{t \supset t} & \text{if } m(s,t) \text{ odd} \end{array}$$
$$1 \otimes \Gamma_{s,t \supset t} \longmapsto \begin{array}{ll} -2(1 \otimes \Gamma_{t \supset t}) & \text{if } m(s,t) \text{ even} \\ -1 \otimes \Gamma_{t \supset t} + 1 \otimes \Gamma_{s \supset s} & \text{if } m(s,t) \text{ odd} \end{array}$$

The kernel of δ_2 is generated by $1 \otimes \Gamma_{s \supset s}$ and $1 \otimes \Gamma_{t \supset t}$. Modding out by the image of δ_3 gives that both of these generators have order two, and when $m(s,t)$ is odd they are identified. This completes the proof. $\square$

Proof of Lemma 2.5.21. We apply the transfer map as defined in Proposition 2.3.15 to the generator(s) of $H_2(W_{\{s,t\}}; \mathbb{Z}_T)$ and then the degree two collapse map f_2 as computed in Section 2.5.8.

For $m(s,t)$ even, consider this map on the generators $1 \otimes \Gamma_{s \supset s}$ and $1 \otimes \Gamma_{t \supset t}$ of $H_2(W_{\{s,t\}}; \mathbb{Z}_T) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in turn, restricted to the summand $H_2(W_s; \mathbb{Z}_s)$ in the image:

$$\begin{aligned} 1 \otimes \Gamma_{s \supset s} &\overset{d^1}{\mapsto} \sum_{\beta \in W_s^T} 1\beta^{-1} \otimes \beta \Gamma_{s \supset s} \\ &= \sum_{l=0}^{m(s,t)-1} (-1)^l \otimes \pi(t,s;l) \Gamma_{s \supset s} \\ &\overset{f_2}{\mapsto} 1 \otimes \Gamma_{s \supset s} - 1 \otimes \Gamma_{s \supset s} \\ &= 0. \end{aligned}$$

When applying the collapse map f_2 above, we note that $\pi(t,s;l)s$ is $(W_s,\emptyset)$-reduced provided $l \notin \{0, m(s,t)-1\}$ and $\pi(t,s;m(s,t)-1)s = \pi(s,t;m(s,t))$ which may be written such that it begins with s. Similarly:

$$\begin{aligned}
1 \otimes \Gamma_{t\supset t} &\overset{d^1}{\mapsto} \sum_{\beta \in W_s^T} 1\beta^{-1} \otimes \beta\Gamma_{t\supset t} \\
&= \sum_{l=0}^{m(s,t)-1} (-1)^l \otimes \pi(t,s;l)\Gamma_{t\supset t} \\
&\overset{f_2}{\mapsto} 0,
\end{aligned}$$

where the final equality is due to the fact that $\pi(t,s;l)t$ is $(W_s,\emptyset)$-reduced for all $0 \le l \le m(s,t)-1$.

Similarly both generators are mapped to zero when restricted to the $H_2(W_t;\mathbb{Z}_t) = \mathbb{Z}_2$ summand in the image.

For $m(s,t)$ odd we have by similar methods the generator $1 \otimes \Gamma_{s\supset s}$ of $H_2(W_{\{s,t\}};\mathbb{Z}_T) = \mathbb{Z}_2$ is mapped as follows:

$$\begin{aligned}
1 \otimes \Gamma_{s\supset s} &\overset{d^1}{\mapsto} \sum_{\beta \in W_s^T} 1\beta^{-1} \otimes \beta\Gamma_{s\supset s} \\
&= \sum_{l=0}^{m(s,t)-1} (-1)^l \otimes \pi(t,s;l)\Gamma_{s\supset s} \\
&\overset{f_2}{\mapsto} 1 \otimes \Gamma_{s\supset s}
\end{aligned}$$

When applying the collapse map f_2 we note that $\pi(t,s;l)s$ is now $(W_s,\emptyset)$-reduced provided $l \neq 0$. Therefore $1 \otimes \Gamma_{s\supset s}$ is mapped to the generator of $H_2(W_s;\mathbb{Z}_s) = \mathbb{Z}_2$. Similarly, since $1 \otimes \Gamma_{s\supset s}$ is identified with $1 \otimes \Gamma_{t\supset t}$ in $H_2(W_{\{s,t\}};\mathbb{Z}_T) = \mathbb{Z}_2$, when restricted to the $H_2(W_t;\mathbb{Z}_t) = \mathbb{Z}_2$ summand in the image, the generator of $H_2(W_{\{s,t\}};\mathbb{Z}_T) = \mathbb{Z}_2$ is also mapped to the generator of $H_2(W_t;\mathbb{Z}_t) = \mathbb{Z}_2$. This completes the proof. □

Proof of Proposition 2.5.24. The twisted resolution for a general Coxeter group with 3 generators, up to degree two, follows from the calculations in Example 2.5.7 and is given below:

$$\mathbb{Z} \underset{W_T}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_T}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_T}{\otimes} C_0$$

Generators:

$1 \otimes \Gamma_{s \supset s}$	$1 \otimes \Gamma_s$	$1 \otimes \Gamma_\emptyset$
$1 \otimes \Gamma_{t \supset t}$	$1 \otimes \Gamma_t$	
$1 \otimes \Gamma_{u \supset u}$	$1 \otimes \Gamma_u$	
$1 \otimes \Gamma_{s,t}$		
$1 \otimes \Gamma_{t,u}$		
$1 \otimes \Gamma_{s,u}$		

Differentials:

$$1 \otimes \Gamma_s \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_t \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_u \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_{s \supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{t \supset t} \longmapsto 0$$

$$1 \otimes \Gamma_{u \supset u} \longmapsto 0$$

$$1 \otimes \Gamma_{s,t} \longmapsto m(s,t)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_t))$$

$$1 \otimes \Gamma_{t,u} \longmapsto m(t,u)((1 \otimes \Gamma_t) - (1 \otimes \Gamma_u))$$

$$1 \otimes \Gamma_{s,u} \longmapsto m(s,u)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_u))$$

The kernel of δ_1 is therefore generated by

$$\alpha = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \text{ and } \beta = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u)$$

and the relations given by the image of δ_2 are:

$$m(s,t)\alpha = 0,\ m(s,u)\beta = 0, \text{ and } m(t,u)(\beta - \alpha) = 0.$$

Applying this to the groups in question gives:

- For $W_T = W(A_3)$:

$$\begin{aligned} 3\alpha = 0,\ 2\beta = 0,\ \text{and } 3(\beta - \alpha) &= 0 \\ \Rightarrow 3\beta &= 0 \\ \Rightarrow \beta &= 0 \end{aligned}$$

which gives $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_3$ generated by α.

- For $W_T = W(B_3)$:

$$\begin{aligned} 4\alpha = 0,\ 2\beta = 0,\ \text{and } 3(\beta - \alpha) &= 0 \\ \Rightarrow -3\alpha &= -3\beta \\ \Rightarrow \alpha &= \beta \end{aligned}$$

which gives $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\alpha = \beta$.

- For $W_T = W(H_3)$:

$$\begin{aligned} 5\alpha = 0,\ 2\beta = 0,\ \text{and } 3(\beta - \alpha) &= 0 \\ \Rightarrow -3\alpha &= -3\beta \\ \Rightarrow 2\alpha &= \beta \\ \Rightarrow 4\alpha &= 2\beta = 0 \\ \Rightarrow \alpha &= 0 \\ \Rightarrow 3\beta &= 0 \\ \Rightarrow \beta &= 0 \end{aligned}$$

which gives $H_1(W_T; \mathbb{Z}_T) = 0$.

- For $W_T = W(I_2(p)) \times W(A_1)$:

$$\begin{aligned} p\alpha = 0,\ 2\beta = 0,\ \text{and } 2(\beta - \alpha) &= 0 \\ \Rightarrow 2\alpha &= 0. \end{aligned}$$

This gives

$$H_1(W_T; \mathbb{Z}_T) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } m(s,t) \text{ is even} \\ \mathbb{Z}_2 & \text{if } m(s,t) \text{ is odd} \end{cases}$$

with generators α and β in the even case, and β in the odd case.

□

Proof of Proposition 2.5.25. Consider the twisted resolution of this group from Example 2.5.7:

$$\mathbb{Z} \underset{W_T}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_T}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_T}{\otimes} C_0$$

Generators:

$$\begin{array}{lll} 1 \otimes \Gamma_{s \supset s} & 1 \otimes \Gamma_s & 1 \otimes \Gamma_\emptyset \\ 1 \otimes \Gamma_{t \supset t} & 1 \otimes \Gamma_t & \\ 1 \otimes \Gamma_{s,t} & & \end{array}$$

Differentials:

$$1 \otimes \Gamma_s \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_t \longmapsto -2(1 \otimes \Gamma_\emptyset)$$

$$1 \otimes \Gamma_{s \supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{t \supset t} \longmapsto 0$$

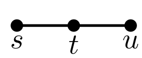
$$1 \otimes \Gamma_{s,t} \longmapsto m(s,t)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_t))$$

Computing the kernel of δ_1 gives generator $\gamma = 1 \otimes \Gamma_s - 1 \otimes \Gamma_t$, and the image of δ_2 gives the relation $m(s,t)\gamma = 0$. This completes the proof.

□

Proof of Proposition 2.5.29. For the finite groups with generating set of size two, the target of the d^1 differential is 0, and so d^1 is the zero map.

For each of the finite groups with generating set $T = \{s, t, u\}$, we apply the transfer and collapse map for each two generator subgroup in turn. This can be calculated by hand, but we do this using Python and the PyCox package [**26**] for the cases $W_T = W(A_3)$ and $W_T = W(B_3)$. The code (given in Appendix A) takes as input a Coxeter group W_T, I a subset of T and w an element of T. It returns the image of $1 \otimes \Gamma_w$ under the transfer and collapse map from $H_1(W_T; \mathbb{Z}_T)$ to $H_1(W, \mathbb{Z}_I)$. A sample example of the code in use is included in Example A.1. The maps are given on the 3-generator subgroups as follows, where below we consider the transfer and collapse map to the three 2-generator subgroups: $I = \{s, t\}$, $I = \{s, u\}$ and $I = \{t, u\}$. For $W_T = W(I_2(p)) \times W(A_1)$ we calculate the differential and collapse by hand.

- $W_T = W(A_3)$ with diagram 

  $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_3 = \langle \alpha \rangle$ has generator $\alpha = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)$.

– $I = \{s, t\}$

$$\begin{aligned} \alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\ &\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t\Big) \\ &\overset{f^1}{\mapsto} (1 \otimes \Gamma_s - 1 \otimes \Gamma_s) - (1 \otimes \Gamma_t - 1 \otimes \Gamma_s) \\ &= 1 \otimes \Gamma_s - 1 \otimes \Gamma_t, \end{aligned}$$

so the image of the generator of $H_1(W_T; \mathbb{Z}_T)$ is the generator of $H_1(W_I; \mathbb{Z}_I) = \mathbb{Z}_3 = \langle 1 \otimes \Gamma_s - 1 \otimes \Gamma_t \rangle$.

– $I = \{s, u\}$

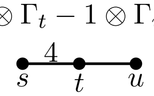
$$\begin{aligned} \alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\ &\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t\Big) \\ &\overset{f^1}{\mapsto} (1 \otimes \Gamma_s + 1 \otimes \Gamma_u) - (1 \otimes \Gamma_u + 1 \otimes \Gamma_s) \\ &= 0, \end{aligned}$$

so the image of the generator of $H_1(W_T; \mathbb{Z}_T)$ is 0.

– $I = \{t, u\}$

$$\begin{aligned} \alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\ &\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t\Big) \\ &\overset{f^1}{\mapsto} (1 \otimes \Gamma_t - 1 \otimes \Gamma_t) - (1 \otimes \Gamma_t - 1 \otimes \Gamma_u) \\ &= 1 \otimes \Gamma_u - 1 \otimes \Gamma_t, \end{aligned}$$

so the image of the generator of $H_1(W_T; \mathbb{Z}_T)$ is minus the generator of $H_1(W_I; \mathbb{Z}_I) = \mathbb{Z}_3 = \langle 1 \otimes \Gamma_t - 1 \otimes \Gamma_u \rangle$.

• $W_T = W(B_3)$ with diagram 

$H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2 = \langle \alpha \rangle$ has generator $\alpha = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)$.

– $I = \{s,t\}$

$$\begin{aligned}
\alpha &= (1\otimes\Gamma_s)-(1\otimes\Gamma_t)\\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_s\Big)-\Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_t\Big)\\
&\overset{f^1}{\mapsto} (1\otimes\Gamma_s-1\otimes\Gamma_s+1\otimes\Gamma_s-1\otimes\Gamma_s)-(1\otimes\Gamma_t-1\otimes\Gamma_t)\\
&= 0,
\end{aligned}$$

so the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

– $I = \{s,u\}$

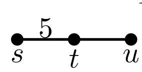
$$\begin{aligned}
\alpha &= (1\otimes\Gamma_s)-(1\otimes\Gamma_t)\\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_s\Big)-\Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_t\Big)\\
&\overset{f^1}{\mapsto} (1\otimes\Gamma_s-1\otimes\Gamma_s+1\otimes\Gamma_s-1\otimes\Gamma_s)-(1\otimes\Gamma_u-1\otimes\Gamma_u)\\
&= 0,
\end{aligned}$$

so the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

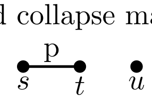
– $I = \{t,u\}$

$$\begin{aligned}
\alpha &= (1\otimes\Gamma_s)-(1\otimes\Gamma_t)\\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_s\Big)-\Big(\sum_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_t\Big)\\
&\overset{f^1}{\mapsto} (0)-(1\otimes\Gamma_t-1\otimes\Gamma_u-1\otimes\Gamma_t+1\otimes\Gamma_u)\\
&= 0,
\end{aligned}$$

so the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

- $W_T = W(H_3)$ with diagram 

  $H_1(W_T;\mathbb{Z}_T)=0$ and so the transfer and collapse map is zero.

- $W_T = W(I_2(p))\times W(A_1)$ with diagram 

  When p is even, $H_1(W_T;\mathbb{Z}_T)=\mathbb{Z}_2\oplus\mathbb{Z}_2$ with generators $\alpha=(1\otimes\Gamma_s)-(1\otimes\Gamma_t)$ and $\beta=(1\otimes\Gamma_s)-(1\otimes\Gamma_u)$. The transfer and collapse maps for each subgroup are therefore:

– $I = \{s, t\}$

$$\begin{aligned}
\alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\
&\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t) \\
&= (1 \otimes \Gamma_s - 1 \otimes u\Gamma_s) - (1 \otimes \Gamma_t - 1 \otimes u\Gamma_t) \\
&\overset{f^1}{\mapsto} (1 \otimes \Gamma_s - 1 \otimes G_s) - (1 \otimes \Gamma_t - 1 \otimes \Gamma_t) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\
&\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u) \\
&= (1 \otimes \Gamma_s - 1 \otimes u\Gamma_s) - (1 \otimes \Gamma_u - 1 \otimes u\Gamma_u) \\
&\overset{f^1}{\mapsto} (1 \otimes \Gamma_s - 1 \otimes \Gamma_s) - (0) \\
&= 0,
\end{aligned}$$

so the image of either generator of $H_1(W_T; \mathbb{Z}_T)$ is 0.

– $I = \{s, u\}$

$$\begin{aligned}
\alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\
&\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t) \\
&= (\sum_{l=0}^{p-1} (-1)^l \otimes \pi(t, s; l)\Gamma_s) - ((-1)^l \otimes \pi(t, s; l)\Gamma_t) \\
&\overset{f^1}{\mapsto} 1 \otimes \Gamma_s + (-1)^{p-1}(1 \otimes \Gamma_s) - 0 \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
\beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u\Big) \\
&= \sum_{l=0}^{p-1} ((-1)^l \otimes \pi(t,s;l)\Gamma_s - (-1)^l \otimes \pi(t,s;l)\Gamma_u) \\
&\overset{f^1}{\mapsto} 1 \otimes \Gamma_s + (-1)^{p-1}(1 \otimes \Gamma_s) - \sum_{l=0}^{p-1} (-1)^l \otimes \Gamma_u \\
&= 0.
\end{aligned}
$$

Here applying f_1, we note that $\pi(t,s;l)s$ is $(I,\emptyset)$-reduced for $l \neq 0, p-1$, $\pi(t,s;l)t$ is $(I,\emptyset)$-reduced for all $0 \leq l \leq p-1$ and $\pi(t,s;l)u = u(\pi(t,s;l))$ for all $0 \leq l \leq p-1$. So the image of either generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

– $I = \{t,u\}$

This case is symmetric to the case $I = \{s,u\}$ and so the image of either generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

When p is odd, $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ with generator $\beta = (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u)$. The transfer and collapse maps for each subgroup are therefore:

– $I = \{s,t\}$

$$
\begin{aligned}
\beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\
&\overset{d^1}{\mapsto} \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s\Big) - \Big(\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u\Big) \\
&= (1 \otimes \Gamma_s - 1 \otimes u\Gamma_s) - (1 \otimes \Gamma_u - 1 \otimes u\Gamma_u) \\
&\overset{f^1}{\mapsto} (1 \otimes \Gamma_s - 1 \otimes \Gamma_s) - (0) \\
&= 0,
\end{aligned}
$$

so the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is 0.

– $I = \{s, u\}$

$$\begin{aligned}
\beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\
&\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u) \\
&= \sum_{l=0}^{p-1}((-1)^l \otimes \pi(t,s;l)\Gamma_s - (-1)^l \otimes \pi(t,s;l)\Gamma_u) \\
&\overset{f^1}{\mapsto} 1 \otimes \Gamma_s - \sum_{l=0}^{p-1}(-1)^l \otimes \Gamma_u \\
&= 1 \otimes \Gamma_s - 1 \otimes \Gamma_u.
\end{aligned}$$

Here applying f_1, we note that $\pi(t,s;l)s$ is $(I,\emptyset)$-reduced for $l \neq 0$ and $\pi(t,s;l)u = u(\pi(t,s;l))$ for all $0 \leq l \leq p-1$. So the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is the generator of $H_1(W_I;\mathbb{Z}_T) = \mathbb{Z}_2$ since $m(s,u) = 2$.

– $I = \{t, u\}$

This case is symmetric to the case $I = \{s, u\}$ and so the image of the generator of $H_1(W_T;\mathbb{Z}_T)$ is the generator of $H_1(W_I;\mathbb{Z}_T) = \mathbb{Z}_2$ since $m(t,u) = 2$.

□

Proof of Lemma 2.5.41. The E^2 page for the Coxeter group $W(A_4)$ is given by

$$\begin{array}{c|cccccc}
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_3 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 &
\end{array}$$

We have the following diagrams for $W = W(A_4)$:

$$D_W: \underset{s}{\bullet}\!-\!\!-\!\underset{t}{\bullet}\!-\!\!-\!\underset{u}{\bullet}\!-\!\!-\!\underset{v}{\bullet} \qquad\qquad \mathcal{D}_{odd} = \mathcal{D}_W$$

$$\mathcal{D}_{\bullet\bullet}: \{s,u\}\bullet\!-\!\!-\!\underset{\{s,v\}}{\bullet}\!-\!\!-\!\bullet\{t,v\} \qquad\qquad \mathcal{D}_{A_2}: \{s,t\}\bullet\!-\!\!-\!\underset{\{t,u\}}{\bullet}\!-\!\!-\!\bullet\{u,v\}$$

$$\mathcal{D}_{A_3}: \{s,t,u\}\bullet\!-\!\!-\!\!-\!\bullet\{t,u,v\} \qquad\qquad \mathcal{D}^{\square}_{\bullet\bullet} = \mathcal{D}_{\bullet\bullet}$$

and $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is the empty diagram. Computing the terms in the spectral sequence as defined at the start of this section therefore gives:

$$H_0(\mathcal{D}_{odd};\mathbb{Z}_2) = \mathbb{Z}_2$$

$$\begin{array}{rl} & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus \big(\bigoplus\limits_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}\big) \\ = & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus 0 \\ = & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \end{array}$$

$$\begin{array}{rl} & H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus \big(\bigoplus\limits_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2\big) \\ = & 0 \oplus 0 \oplus \mathbb{Z}_2 \oplus 0 \\ = & \mathbb{Z}_2 \end{array}$$

$\square$

Proof of Lemma 2.5.43. The twisted resolution for a finite Coxeter group with 4 generators, up to degree two, easily follows from the calculations in Example 2.5.7 and is as follows, where in the diagram below $x \in \{s,t,u,v\}$:

$$\mathbb{Z} \underset{W_T}{\otimes} C_2 \xrightarrow{\delta_2} \mathbb{Z} \underset{W_T}{\otimes} C_1 \xrightarrow{\delta_1} \mathbb{Z} \underset{W_T}{\otimes} C_0$$

Generators:

$1 \otimes \Gamma_{x \supset x}$	$1 \otimes \Gamma_x$	$1 \otimes \Gamma_\emptyset$
$1 \otimes \Gamma_{s,t}$		
$1 \otimes \Gamma_{t,u}$		
$1 \otimes \Gamma_{u,v}$		
$1 \otimes \Gamma_{s,u}$		
$1 \otimes \Gamma_{s,v}$		
$1 \otimes \Gamma_{t,v}$		

Differentials: $1 \otimes \Gamma_x \longmapsto -2(1 \otimes \Gamma_\emptyset)$

$$\begin{aligned}
1 \otimes \Gamma_{x \supset x} &\longmapsto 0 \\
1 \otimes \Gamma_{s,t} &\longmapsto m(s,t)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)) \\
1 \otimes \Gamma_{t,u} &\longmapsto m(t,u)((1 \otimes \Gamma_t) - (1 \otimes \Gamma_u)) \\
1 \otimes \Gamma_{u,v} &\longmapsto m(u,v)((1 \otimes \Gamma_u) - (1 \otimes \Gamma_v)) \\
1 \otimes \Gamma_{s,u} &\longmapsto m(s,u)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_u)) \\
1 \otimes \Gamma_{s,v} &\longmapsto m(s,v)((1 \otimes \Gamma_s) - (1 \otimes \Gamma_v)) \\
1 \otimes \Gamma_{t,v} &\longmapsto m(t,v)((1 \otimes \Gamma_t) - (1 \otimes \Gamma_v))
\end{aligned}$$

The kernel of δ_2 is therefore generated by

$$\begin{aligned}
\alpha &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)\ , \\
\beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u), \\
\gamma &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_v),
\end{aligned}$$

and the relations given by the image of δ_3 are:

$$\begin{aligned}
m(s,t)\alpha &= 0 & m(t,u)(\beta - \alpha) &= 0 \\
m(s,u)\beta &= 0 & m(t,v)(\gamma - \alpha) &= 0 \\
m(s,v)\gamma &= 0 & m(u,v)(\gamma - \beta) &= 0.
\end{aligned}$$

Applying this to the groups in question gives:

- For $W_T = W(A_4)$:

$$\begin{array}{rclrclrclrcl} 3\alpha &=& 0 & 3(\beta-\alpha) &=& 0 &\Rightarrow& 3\beta &=& 0 &\Rightarrow& \beta = 0 \\ 2\beta &=& 0 & 2(\gamma-\alpha) &=& 0 &\Rightarrow& -2\alpha &=& 0 &\Rightarrow& \alpha = 0 \\ 2\gamma &=& 0 & 3(\gamma-\beta) &=& 0 &\Rightarrow& 3\gamma &=& 0 &\Rightarrow& \gamma = 0 \end{array}$$

which gives $H_1(W_T;\mathbb{Z}_T) = 0$.

- For $W_T = W(B_4)$:

$$\begin{array}{rclrclrclrcl} 4\alpha &=& 0 & 3(\beta-\alpha) &=& 0 &\Rightarrow& \beta-\alpha &=& 0 &\Rightarrow& \beta = \alpha \\ 2\beta &=& 0 & 2(\gamma-\alpha) &=& 0 & & & & & & \\ 2\gamma &=& 0 & 3(\gamma-\beta) &=& 0 &\Rightarrow& \gamma-\beta &=& 0 &\Rightarrow& \gamma = \beta \end{array}$$

which gives $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\alpha = \beta = \gamma$.

- For $W_T = W(H_4)$:

$$\begin{array}{rcl} 5\alpha &=& 0 \\ 2\beta &=& 0 \\ 2\gamma &=& 0 \end{array} \Rightarrow \begin{array}{rclrclrcl} 3(\beta-\alpha) &=& 0 &\Rightarrow& \beta-2\alpha &=& 0 &\Rightarrow& \beta = 2\alpha \\ 4\alpha &=& 0 &\Rightarrow& \alpha &=& 0 &\Rightarrow& \beta = 0 \\ 2(\gamma-\alpha) &=& 0 & & & & & & \\ 3(\gamma-\beta) &=& 0 &\Rightarrow& \gamma &=& 0 & & \end{array}$$

which gives $H_1(W_T;\mathbb{Z}_T) = 0$.

- For $W_T = W(F_4)$:

$$\begin{array}{rclrclrclrcl} 3\alpha &=& 0 & 4(\beta-\alpha) &=& 0 &\Rightarrow& 4\alpha &=& 0 &\Rightarrow& \alpha = 0 \\ 2\beta &=& 0 & 2(\gamma-\alpha) &=& 0 &\Rightarrow& 2\gamma &=& 0 & & \\ 2\gamma &=& 0 & 3(\gamma-\beta) &=& 0 &\Rightarrow& \gamma &=& \beta & & \end{array}$$

which gives $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\beta = \gamma$.

- For $W_T = W(D_4)$:

$$\begin{array}{rclrclrcl} 2\alpha &=& 0 & 3(\beta-\alpha) &=& 0 &\Rightarrow \alpha &=& 0 \\ 3\beta &=& 0 & 2(\gamma-\alpha) &=& 0 & & & \\ 2\gamma &=& 0 & 3(\gamma-\beta) &=& 0 &\Rightarrow \gamma &=& 0 \end{array}$$

which gives $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_3$ generated by β.

- For $W_T = W(I_2(p)) \times W(I_2(q))$:

$$\begin{array}{rclrclrcl} p\alpha &=& 0 & 2(\beta-\alpha) &=& 0 &\Rightarrow 2\alpha &=& 0 \\ 2\beta &=& 0 & 2(\gamma-\alpha) &=& 0 & & & \\ 2\gamma &=& 0 & q(\gamma-\beta) &=& 0 & & & \end{array}$$

This gives

$$H_1(W_T;\mathbb{Z}_T) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ and } q \text{ are both even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ is odd and } q \text{ is even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ is even and } q \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p \text{ and } q \text{ are both odd} \end{cases}$$

with generators α when p even and β, γ when q even, with $\gamma = \beta$ when q odd and $\alpha = 0$ when p odd.

When we have the product of a finite group with 3 generators and $W(A_1)$, the generators and relations become as follows:

$$\begin{array}{rclcrclcrcl} m(s,t)\alpha &=& 0 & \quad & m(t,u)(\beta-\alpha) &=& 0 & & & & \\ m(s,u)\beta &=& 0 & & 2(\gamma-\alpha) &=& 0 & \Rightarrow & 2\alpha &=& 0 \\ 2\gamma &=& 0 & & 2(\gamma-\beta) &=& 0 & \Rightarrow & 2\beta &=& 0 \end{array}$$

so given the generators and relations in the first homology of the 3-generator subgroup we can calculate the homology of the product with $W(A_1)$ by:

- adding an extra $\mathbb{Z}_2$ summand generated by γ
- adding the relations $2\alpha = 0$ and 2β=0.

Applying this to the 3 generator groups from Proposition 2.5.24 gives the following results:

- For $W_T = W(A_3) \times W(A_1)$: $H_1(W(A_3); \mathbb{Z}_T) = \mathbb{Z}_3$ generated by α. Adding the $\mathbb{Z}_2$ summand generated by γ and the relation $2\alpha = 0$ gives $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by γ.
- For $W_T = W(B_3) \times W(A_1)$: $H_1(B_3; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\alpha = \beta$. Adding the $\mathbb{Z}_2$ summand generated by γ and the relations $2\alpha = 2\beta = 0$ gives $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $\alpha = \beta$ and γ.
- For $W_T = W(H_3) \times W(A_1)$: $H_1(W(H_3); \mathbb{Z}_T) = 0$. Adding the $\mathbb{Z}_2$ summand generated by γ gives $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by γ.

□

Proof of Lemma 2.5.44. For each possible 4 generator subgroup W_T, we let I cycle through the subsets of T of size 3 and consider transfer and collapse maps from W_T to W_I:

- For $W_T = W(A_4)$: $H_1(W_T; \mathbb{Z}_T) = 0$, so all maps are zero.
- For $W_T = W(B_4)$: $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\alpha = \beta = \gamma$.
  - $I = \{s, t, u\}$

$$\begin{array}{rcl} \alpha &=& (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t) \\ & \stackrel{d^1}{\mapsto} & (\sum\limits_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum\limits_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_t) \\ & \stackrel{f^1}{\mapsto} & 0 \end{array}$$

  - $I = \{s, t, v\}$ Similarly, $\alpha \mapsto 0$.
  - $I = \{s, u, v\}$ Similarly, $\alpha \mapsto 0$.
  - $I = \{t, u, v\}$ Similarly, $\alpha \mapsto 0$.
- For $W_T = W(H_4)$: $H_1(W_T; \mathbb{Z}_T) = 0$ so all maps are zero.
- For $W_T = W(F_4)$: $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ generated by $\beta = \gamma$.

– $I = \{s, t, u\}$

$$\begin{aligned} \beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\ &\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u) \\ &\overset{f^1}{\mapsto} 0 \end{aligned}$$

– $I = \{s, t, v\}$ Similarly, $\beta \mapsto 0$.
– $I = \{s, u, v\}$ Similarly, $\beta \mapsto 0$.
– $I = \{t, u, v\}$ Similarly, $\beta \mapsto 0$.

- For $W_T = W(D_4)$: $H_1(W_T; \mathbb{Z}_T) = \mathbb{Z}_3$ generated by β.
  – $I = \{s, t, u\}$

$$\begin{aligned} \beta &= (1 \otimes \Gamma_s) - (1 \otimes \Gamma_u) \\ &\overset{d^1}{\mapsto} (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_s) - (\sum_{\beta \in W_I^T} 1\beta^{-1} \otimes \beta\Gamma_u) \\ &\overset{f^1}{\mapsto} 0 - (2(1 \otimes \Gamma_u) - (1 \otimes \Gamma_s) - (1 \otimes \Gamma_t)) \\ &= 1 \otimes \Gamma_s + 1 \otimes \Gamma_t - 2(1 \otimes \Gamma_u) \\ &= 2(1 \otimes \Gamma_s - 1 \otimes \Gamma_u) - (1 \otimes \Gamma_s - 1 \otimes \Gamma_t) \end{aligned}$$

  The generator for $H_1(W_I; \mathbb{Z}_I) = \mathbb{Z}_3$ when $I = \{s, t, u\}$ is $(1 \otimes \Gamma_s - 1 \otimes \Gamma_u)$ and in this homology group $(1 \otimes \Gamma_s - 1 \otimes \Gamma_t)$ is identified with zero. Therefore the generator for $H_1(W_T; \mathbb{Z}_T)$ gets mapped to 2 times the generator of $H_1(W_I; \mathbb{Z}_T)$ when $I = \{s, t, u\}$.
  – $I = \{s, t, v\}$ In this case a similar computation gives $\alpha \mapsto 0$.
  – $I = \{s, u, v\}$ This case is symmetric to that of $I = \{s, t, u\}$. Therefore the generator for $H_1(W_T; \mathbb{Z}_T)$ gets mapped to 2 times the generator of $H_1(W_I; \mathbb{Z}_T)$ when $I = \{s, u, v\}$.
  – $I = \{t, u, v\}$ This case is symmetric to that of $I = \{s, t, u\}$. Therefore the generator for $H_1(W_T; \mathbb{Z}_T)$ gets mapped to 2 times the generator of $H_1(W_I; \mathbb{Z}_T)$ when $I = \{t, u, v\}$.
- For $W_T = W(I_2(p)) \times W(I_2(q))$:

$$H_1(W_T; \mathbb{Z}_T) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ and } q \text{ are both even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ is odd and } q \text{ is even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \text{ is even and } q \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p \text{ and } q \text{ are both odd} \end{cases}$$

with generators α when p even and β, γ when q even, with $\gamma = \beta$ when q odd and $\alpha = 0$ when p odd. By symmetry, we only need to compute the transfer and collapse map for $I = \{s,t,u\}$ in the 4 cases that either p and q are both odd, both even, p is odd and q is even, or p is even and q is odd.

- p and q are both odd: by similar reasoning to Proposition 2.5.29, it follows generator β maps as the identity to the generator of $H_1(W_I;\mathbb{Z}_I)$.
- p and q are both even: all generators are mapped to zero.
- p is odd and q is even: both generators β and γ are mapped to zero.
- p is even and q is odd: both generators α and $\beta = \gamma$ are mapped as the identity to the two generators of $H_1(W_I;\mathbb{Z}_I)$.

- For $W_T = W(A_3) \times W(A_1)$: $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ generated by γ.
  - $I = \{s,t,u\}$

$$\begin{array}{rcl}\gamma & = & (1\otimes\Gamma_s) - (1\otimes\Gamma_v)\\ & \stackrel{d^1}{\mapsto} & (\sum\limits_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_s) - (\sum\limits_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_t)\\ & \stackrel{f^1}{\mapsto} & 0\end{array}$$

  - $I = \{s,t,v\}$ Similarly, $\gamma \mapsto 0$.
  - $I = \{s,u,v\}$

$$\begin{array}{rcl}\gamma & = & (1\otimes\Gamma_s) - (1\otimes\Gamma_v)\\ & \stackrel{d^1}{\mapsto} & (\sum\limits_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_s) - (\sum\limits_{\beta\in W_I^T} 1\beta^{-1}\otimes\beta\Gamma_t)\\ & \stackrel{f^1}{\mapsto} & 1\otimes\Gamma_s + 1\otimes\Gamma_u - 2(1\otimes\Gamma_v)\\ & = & 2(1\otimes\Gamma_s - 1\otimes\Gamma_v) - (1\otimes\Gamma_s - 1\otimes\Gamma_u)\end{array}$$

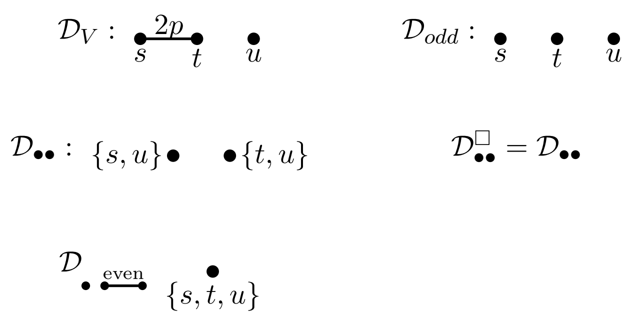
    The generators for $H_1(W_I;\mathbb{Z}_T)$ when $I = \{t,u,v\}$ are $(1\otimes\Gamma_t - 1\otimes\Gamma_u)$ and $(1\otimes\Gamma_t - 1\otimes\Gamma_v)$ and they both generate a $\mathbb{Z}_2$ summand. Therefore the generator for $H_1(W_T;\mathbb{Z}_T)$ gets mapped to the generator $1\otimes\Gamma_t - 1\otimes\Gamma_v$ of $H_1(W_I;\mathbb{Z}_I)$ when $I = \{s,u,v\}$.
  - $I = \{t,u,v\}$ $\alpha \mapsto 0$
- For $W_T = W(B_3)\times W(A_1)$: $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2\oplus\mathbb{Z}_2$ generated by $\alpha = \beta$ and γ. Using the Python script in Appendix A we compute that transfer maps to all 3 generator subgroups are 0.
- For $W_T = W(H_3) \times W(A_1)$: $H_1(W_T;\mathbb{Z}_T) = \mathbb{Z}_2$ generated by γ. Using the Python script in Appendix A we compute that transfer maps to all 3 generator subgroups are 0.

□

Proof of Lemma 2.5.50. The E^∞ page for the Coxeter group $V = W(I_2(2p)) \times W(A_1)$, for $p > 1$ is given by

$$\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}$$

We have the following diagrams for $V = W(I_2(2p)) \times W(A_1)$, when $p > 1$:



where $\mathcal{D}_{A_2}$ and $\mathcal{D}_{A_3}$ are the empty diagram. Below we compute the terms in the spectral sequence given at the start of this section:

$$H_0(\mathcal{D}_{odd}; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\begin{aligned}
& H_0(\mathcal{D}_{\bullet\bullet}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2}; \mathbb{Z}_3) \oplus \Big(\bigoplus_{m(s,t)>3, \neq\infty} \mathbb{Z}_{m(s,t)}\Big) \\
= \;& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_{2p} \\
= \;& \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}
\end{aligned}$$

$$\begin{aligned}
& H_1(\mathcal{D}^{\square}_{\bullet\bullet}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\,\overset{\text{even}}{\bullet\!-\!\bullet}}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3}; \mathbb{Z}_2) \oplus \Big(\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2\Big) \\
= \;& 0 \oplus \mathbb{Z}_2 \oplus 0 \oplus 0 \\
= \;& \mathbb{Z}_2.
\end{aligned}$$

The third integral homology of $V = W(I_2(2p)) \times W(A_1)$ can be computed via the Künneth formula for groups, as follows:

$$\begin{aligned}
H_3(W(I_2(2p)) \times W(A_1); \mathbb{Z}) &= \bigoplus_{i+j=3} H(_i(W(I_2(2p)); \mathbb{Z}) \otimes H_j(W(A_1); \mathbb{Z})) \\
&\quad \bigoplus_{i+j=2} Tor(H_i(W(I_2(2p)); \mathbb{Z}), H_j(W(A_1); \mathbb{Z})) \\
&= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}_2 \oplus Z_2
\end{aligned}$$

where we compute this using the following

$$\begin{aligned}
H_0(W(I_2(2p)); \mathbb{Z}) \otimes H_3(W(A_1); \mathbb{Z}) &= \mathbb{Z} \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \\
H_1(W(I_2(2p)); \mathbb{Z}) \otimes H_2(W(A_1); \mathbb{Z}) &= (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \otimes 0 = 0 \\
H_2(W(I_2(2p)); \mathbb{Z}) \otimes H_1(W(A_1); \mathbb{Z}) &= \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \\
H_3(W(I_2(2p)); \mathbb{Z}) \otimes H_0(W(A_1); \mathbb{Z}) &= (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}) \otimes \mathbb{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} \\
Tor(H_0(W(I_2(2p)); \mathbb{Z}), H_2(A_1; \mathbb{Z})) &= Tor(\mathbb{Z}, 0) = 0 \\
Tor(H_1(W(I_2(2p)); \mathbb{Z}), H_1(W(A_1); \mathbb{Z})) &= Tor((\mathbb{Z}_2 \oplus \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2 \oplus Z_2 \\
Tor(H_2(W(I_2(2p)); \mathbb{Z}), H_0(W(A_1); \mathbb{Z})) &= Tor(\mathbb{Z}_2, \mathbb{Z}) = 0
\end{aligned}$$

For the case $p = 1$, i.e. $V = W(I_2(p)) \times W(A_1) = W(A_1) \times W(A_1) \times W(A_1)$, we have the following E^∞ page:

$$\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}$$

Computed via the following diagrams for $V = W(A_1) \times W(A_1) \times W(A_1)$:

$\mathcal{D}_V$: $\underset{s}{\bullet}$ $\underset{t}{\bullet}$ $\underset{u}{\bullet}$ $\qquad$ $\mathcal{D}_{odd}$: $\underset{s}{\bullet}$ $\underset{t}{\bullet}$ $\underset{u}{\bullet}$

$\mathcal{D}_{\bullet\bullet}$: $\{s,t\}\bullet$ $\underset{\{s,u\}}{\bullet}$ $\bullet\{t,u\}$ $\qquad$ $\mathcal{D}^{\square}_{\bullet\bullet} = \mathcal{D}_{\bullet\bullet}$

$\mathcal{D}_{\bullet \overset{\text{even}}{\bullet\!-\!\bullet}}$: $\underset{\{s,t,u\}}{\bullet}$

where $\mathcal{D}_{A_2}$ and $\mathcal{D}_{A_3}$ are the empty diagram. The terms in the spectral sequence as given at the start of this section are therefore:

$$H_0(\mathcal{D}_{odd};\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\begin{aligned} & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \\ = \ & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 0 \oplus 0 \\ = \ & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{aligned}$$

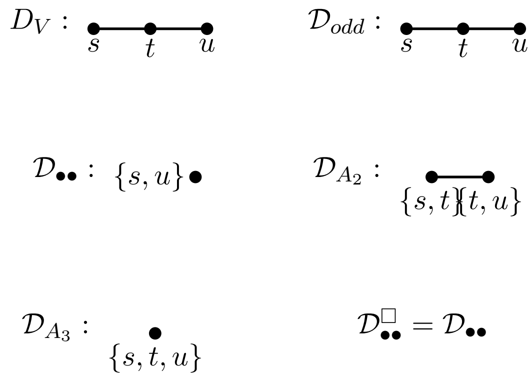
$$\begin{aligned} & H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \\ = \ & 0 \oplus \mathbb{Z}_2 \oplus 0 \oplus 0 \\ = \ & \mathbb{Z}_2 \end{aligned}$$

$\square$

PROOF OF LEMMA 2.5.51. The E^∞ page for the Coxeter group $V = W(A_3)$ is given by

$\vdots$	$\vdots$					
3	0	$\cdots$				
2	0	$\mathbb{Z}_2$	?	$\cdots$		
1	0	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	?		
0	$\mathbb{Z}$	?	?	$\mathbb{Z}_2$	?	$\cdots$
	0	1	2	3	4	$\cdots$

which is computed via the following diagrams for $V = W(A_3)$:



and $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ is the empty diagram. Computing the terms in the spectral sequence therefore gives:

$$H_0(\mathcal{D}_{odd};\mathbb{Z}_2) = \mathbb{Z}_2$$

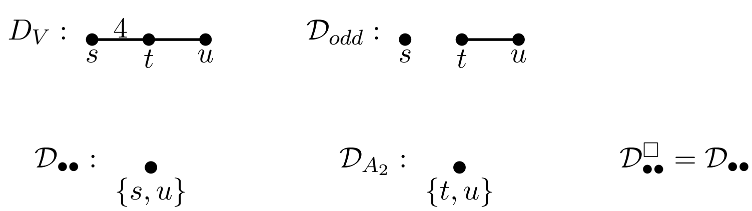
$$\begin{aligned} & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \\ = \; & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus 0 \\ = \; & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \end{aligned}$$

$$\begin{aligned} & H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus (\bigoplus_{\substack{W(H_3)\subseteq W \\ W(B_3)\subseteq W}} \mathbb{Z}_2) \\ = \; & 0 \oplus 0 \oplus \mathbb{Z}_2 \oplus 0 \\ = \; & \mathbb{Z}_2 \end{aligned}$$

$\square$

Proof of Lemma 2.5.55. We compare the spectral sequence for the groups $W(B_3)$ and $W(H_3)$ with their third integral homologies computed using the De Concini - Salvetti resolution.

- For $V = W(B_3)$ the Coxeter group of type B_3 the diagrams are



  and $\mathcal{D}_{A_3}$ and $\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}}$ are the empty diagram. So the entries in the spectral sequence become

$$H_0(\mathcal{D}_{odd};\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

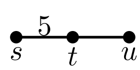
$$
\begin{aligned}
& H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \\
= \; & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \\
= \; & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4
\end{aligned}
$$

$$
\begin{aligned}
& H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\ \bullet\overset{\text{even}}{-}\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus (\bigoplus_{\substack{W(H_3)\subseteq V\\ W(B_3)\subseteq V}} \mathbb{Z}_2) \\
= \; & 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}_2 \\
= \; & \mathbb{Z}_2
\end{aligned}
$$

The E^∞ page for the Coxeter group $V = W(B_3)$ is therefore given by

$$
\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}
$$

- For $V = W(H_3)$ the Coxeter group of type H_3 the diagrams are

D_V : 

$\mathcal{D}_{odd} := \mathcal{D}_W$

$\mathcal{D}_{\bullet\bullet}$: $\bullet$ $\{s, u\}$ $\quad$ $\mathcal{D}_{A_2}$: $\bullet$ $\{t, u\}$ $\quad$ $\mathcal{D}^{\square}_{\bullet\bullet} = \mathcal{D}_{\bullet\bullet}$

and $\mathcal{D}_{A_3}$ and $\mathcal{D}_{\bullet\ \bullet\overset{\text{even}}{-}\bullet}$ are the empty diagram. So the entries in the spectral sequence become

$$H_0(\mathcal{D}_{odd};\mathbb{Z}_2) = \mathbb{Z}_2$$

$$
\begin{aligned}
& H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3) \oplus (\bigoplus_{m(s,t)>3,\neq\infty} \mathbb{Z}_{m(s,t)}) \\
= \; & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5
\end{aligned}
$$

$$\begin{aligned}
& H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\ \overset{\text{even}}{\bullet\!-\!\bullet}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{A_3};\mathbb{Z}_2) \oplus \Big(\bigoplus_{\substack{W(H_3)\subseteq V \\ W(B_3)\subseteq V}} \mathbb{Z}_2\Big) \\
= \quad & 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}_2 \\
= \quad & \mathbb{Z}_2
\end{aligned}$$

The E^∞ page for the Coxeter group $V = W(H_3)$ is therefore given by

$$\begin{array}{c|cccccc}
\vdots & \vdots & & & & & \\
3 & 0 & \cdots & & & & \\
2 & 0 & \mathbb{Z}_2 & ? & \cdots & & \\
1 & 0 & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 & ? & & \\
0 & \mathbb{Z} & ? & ? & \mathbb{Z}_2 & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots
\end{array}$$

To compute the third integral homology of $W(B_3)$ and $W(H_3)$ we use the De Concini - Salvetti resolution from [**18**], with integer coefficients and a trivial action of the group. We expand on Example 2.5.5 to compute the typical resolution of a three generator Coxeter group before tensoring. We let x and y be such that $x, y \in \{s, t, u\}$ with $x < y$ in the ordering.

$$C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2}$$

Generators:

$$\begin{array}{ll} \Gamma_{x\supset x\supset x} & \Gamma_{x\supset x} \\ \Gamma_{x,y\supset x} & \Gamma_{x,y} \\ \Gamma_{x,y\supset t} & \\ \Gamma_{s,t,u} & \end{array}$$

Differentials:

$$\Gamma_{x\supset x\supset x} \longmapsto (x-1)\Gamma_{x\supset x}$$

$$\Gamma_{x,y\supset x} \longmapsto \begin{array}{ll} (1-p(y,x;m(x,y)-1))\Gamma_{x\supset x} - (1+x)\Gamma_{xy} & \text{if } m(x,y) \text{ even} \\ \Gamma_{x\supset x} - p(x,y;m(x,y)-1)\Gamma_{y\supset y} - (1+x)\Gamma_{xy} & \text{if } m(x,y) \text{ odd} \end{array}$$

$$\Gamma_{x,y\supset y} \longmapsto \begin{array}{ll} (-1+p(x,y;m(x,y)-1))\Gamma_{y\supset y} - (1+y)\Gamma_{xy} & \text{if } m(x,y) \text{ even} \\ -\Gamma_{y\supset y} + p(y,x;m(x,y)-1)\Gamma_{x\supset x} - (1+y)\Gamma_{xy} & \text{if } m(x,y) \text{ odd} \end{array}$$

$$\Gamma_{s,t,u} \longmapsto \text{see below}$$

We recall the formula for $\delta_k(e(\Gamma))$ from Equation (4).

$$\delta_k(e(\Gamma)) = \sum_{\substack{i\geq 1 \\ |\Gamma_i|>|\Gamma_{i+1}|}} \sum_{\tau\in\Gamma_i} \sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma')$$

Then $\delta_3(\Gamma_{s,t,u})$ is computed as follows:

$$\begin{aligned} \delta_3(\Gamma_{s,t,u}) &= \sum_{i=1} \sum_{\tau=s,t,u} \sum_{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}} (-1)^{\ell(\beta)+\mu(\{s,t,u\},\tau)}\beta e(\Gamma') \\ &= \sum_{\beta\in W_{\{s,t,u\}}^{\{t,u\}}} (-1)^{\ell(\beta)+1}\beta\Gamma_{t,u} + \sum_{\beta\in W_{\{s,t,u\}}^{\{s,u\}}} (-1)^{\ell(\beta)+2}\beta\Gamma_{s,u} \\ &\quad + \sum_{\beta\in W_{\{s,t,u\}}^{\{s,t\}}} (-1)^{\ell(\beta)+3}\beta\Gamma_{s,t} \end{aligned}$$

$$\begin{aligned} \alpha(\Gamma_{s,t,u},1,\tau,p(s,t;j)) &= i\cdot\ell(\beta) + \sum_{k=1}^{i-1}|\Gamma_k| + \mu(\{s,t,u\},\tau) \\ &= \ell(\beta) + 0 + \mu(\{s,t,u\},\tau) \\ &= \ell(\beta) + \mu(\{s,t,u\},\tau) \end{aligned}$$

Tensoring with the integers over the trivial group action gives the following:

$$\mathbb{Z} \otimes_{W_T} C_3 \xrightarrow{\delta_3} \mathbb{Z} \otimes_{W_T} C_2 \xrightarrow{\delta_2}$$

Differentials:

$$1 \otimes \Gamma_{x\supset x\supset x} \longmapsto 0$$

$$1 \otimes \Gamma_{x,y\supset x} \longmapsto \begin{cases} -2(1 \otimes \Gamma_{xy}) & \text{if } m(x,y) \text{ even} \\ 1 \otimes \Gamma_{x\supset x} - 1 \otimes \Gamma_{y\supset y} - 2(1 \otimes \Gamma_{xy}) & \text{if } m(x,y) \text{ odd} \end{cases}$$

$$1 \otimes \Gamma_{x,y\supset y} \longmapsto \begin{cases} -2(1 \otimes \Gamma_{xy}) & \text{if } m(x,y) \text{ even} \\ -1 \otimes \Gamma_{y\supset y} + 1 \otimes \Gamma_{x\supset x} - 2(1 \otimes \Gamma_{xy}) & \text{if } m(x,y) \text{ odd} \end{cases}$$

We also must compute the differentials mapping in from C_4. In the diagram below we let x and y be such that $x, y \in \{s, t, u\}$ with $x < y$ in the ordering. The differentials that we have not already computed previously are computed below the diagram.

$$C_4 \xrightarrow{\delta_4} C_3 \xrightarrow{\delta_3} C_2$$

Generators:

$\Gamma_{x\supset x\supset x\supset x}$

$\Gamma_{x,y\supset x,y}$

$\Gamma_{x,y\supset x\supset x}$

$\Gamma_{x,y\supset y\supset y}$

$\Gamma_{s,t,u\supset x}$

Differentials:

$$\Gamma_{x\supset x\supset x\supset x} \longmapsto \Gamma_{x\supset x\supset x} + x\Gamma_{x\supset x\supset x}$$

The $\delta_4(\Gamma_{x,y\supset x,y})$ computation is given by:

$$\begin{aligned}
\delta_4(\Gamma_{x,y\supset x,y}) &= \sum_{i=2}\sum_{\tau=x,y}\sum_{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}}(-1)^{\mu(\{x,y\},\tau)}\beta e(\Gamma') \\
&= \sum_{\beta\in W_{\{x,y\}}^{\{y\}}}(-1)^1\beta\Gamma_{x,y\supset y} + \sum_{\beta\in W_{\{x,y\}}^{\{x\}}}(-1)^2\beta\Gamma_{x,y\supset x} \\
&= -(\sum_{\beta\in W_{\{x,y\}}^{\{y\}}}\beta\Gamma_{x,y\supset y}) + \sum_{\beta\in W_{\{x,y\}}^{\{x\}}}\beta\Gamma_{x,y\supset x}
\end{aligned}$$

$$\begin{aligned}
\alpha(\Gamma_{x,y\supset x,y},2,\tau,\beta) &= i\cdot\ell(\beta)+\sum_{k=1}^{i-1}|\Gamma_k|+\mu(\{x,y\},\tau) \\
&= 2\ell(\beta)+2+\mu(\{x,y\},\tau) \\
&= 2\ell(\beta)+2+\mu(\{x,y\},\tau)
\end{aligned}$$

The $\delta_4(\Gamma_{x,y\supset x\supset x})$ computation is similar to that of $\delta_3(\Gamma_{x,y\supset x})$:

$$\delta_3(\Gamma_{s,t\supset s\supset s}) = \begin{cases} \Gamma_{x\supset x\supset x} - p(y,x;m(x,y)-1)\Gamma_{x\supset x\supset x} + \Gamma_{xy\supset x} - x\Gamma_{xy\supset x} & m(x,y) \text{ even} \\ \Gamma_{x\supset x\supset x} - p(x,y;m(x,y)-1)\Gamma_{t\supset t\supset y} + \Gamma_{xy\supset x} - x\Gamma_{xy\supset x} & m(x,y) \text{ odd} \end{cases}$$

The $\delta_4(\Gamma_{x,y\supset y\supset y})$ computation is similar to that of $\delta_3(\Gamma_{x,y\supset y})$:

$$\delta_3(\Gamma_{s,t\supset s\supset s}) = \begin{cases} (-1+p(x,y;m(x,y)-1))\Gamma_{y\supset y\supset y} + (1-y)\Gamma_{xy\supset y} & m(x,y) \text{ even} \\ -\Gamma_{y\supset y\supset y} + p(y,x;m(x,y)-1)\Gamma_{x\supset x\supset x} + (1-y)\Gamma_{xy\supset y} & m(x,y) \text{ odd} \end{cases}$$

The differentials $\delta_4(\Gamma_{s,t,u\supset x})$ with $x\in\{s,t,u\}$ will be computed on a case by case basis for Coxeter groups $W(B_3)$ and $W(H_3)$. Tensoring with $\mathbb{Z}$ gives the following resolution, when again x and y are such that $x,y\in\{s,t,u\}$ with $x<y$ in the ordering.:

$$\mathbb{Z} \otimes_W C_4 \xrightarrow{\delta_4} \mathbb{Z} \otimes_W C_3 \xrightarrow{\delta_3} \mathbb{Z} \otimes_W C_2$$

Differentials:

$$1 \otimes \Gamma_{x\supset x\supset x\supset x} \longmapsto 2(1 \otimes \Gamma_{x\supset x\supset x})$$

$$1 \otimes \Gamma_{x,y\supset x,y} \longmapsto -m(x,y)(1 \otimes \Gamma_{x,y\supset y}) + m(x,y)(1 \otimes \Gamma_{x,y\supset x})$$

$$1 \otimes \Gamma_{x,y\supset x\supset x} \longmapsto \begin{cases} 0 & \text{if } m(x,y) \text{ even} \\ 1 \otimes \Gamma_{x\supset x\supset x} - 1 \otimes \Gamma_{y\supset y\supset y} & \text{if } m(x,y) \text{ odd} \end{cases}$$

$$1 \otimes \Gamma_{x,y\supset y\supset y} \longmapsto \begin{cases} 0 & \text{if } m(x,y) \text{ even} \\ -1 \otimes \Gamma_{y\supset y\supset y} + 1 \otimes \Gamma_{x\supset x\supset x} & \text{if } m(x,y) \text{ odd} \end{cases}$$

For $W(B_3)$ this gives the following resolution, where the computations for $\delta_3(\Gamma_{s,t,u})$ and $\delta_4(\Gamma_{s,t,u\supset x})$ with $x \in \{s,t,u\}$ are given afterwards:

$$\mathbb{Z} \otimes_{W_T} C_3 \xrightarrow{\delta_3} \mathbb{Z} \otimes_{W_T} C_2 \xrightarrow{\delta_2}$$

Differentials:

$$1 \otimes \Gamma_{s\supset s\supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{t\supset t\supset t} \longmapsto 0$$

$$1 \otimes \Gamma_{u\supset u\supset u} \longmapsto 0$$

$$1 \otimes \Gamma_{s,t\supset s} \longmapsto -2(1 \otimes \Gamma_{s,t})$$

$$1 \otimes \Gamma_{s,t\supset t} \longmapsto -2(1 \otimes \Gamma_{s,t})$$

$$1 \otimes \Gamma_{s,u\supset s} \longmapsto -2(1 \otimes \Gamma_{s,u})$$

$$1 \otimes \Gamma_{s,u\supset u} \longmapsto -2(1 \otimes \Gamma_{s,u})$$

$$1 \otimes \Gamma_{t,u\supset t} \longmapsto 1 \otimes \Gamma_{t\supset t} - 1 \otimes \Gamma_{u\supset u} - 2(1 \otimes \Gamma_{t,u})$$

$$1 \otimes \Gamma_{t,u\supset u} \longmapsto -1 \otimes \Gamma_{u\supset u} + 1 \otimes \Gamma_{t\supset t} - 2(1 \otimes \Gamma_{t,u})$$

$$1 \otimes \Gamma_{s,t,u} \longmapsto 0$$

$$\mathbb{Z}\otimes_W C_4 \xrightarrow{\delta_4} \mathbb{Z}\otimes_W C_3 \xrightarrow{\delta_3} \mathbb{Z}\otimes_W C_2$$

Differentials:

$$1\otimes\Gamma_{s\supset s\supset s\supset s} \longmapsto 2(1\otimes\Gamma_{s\supset s\supset s})$$

$$1\otimes\Gamma_{t\supset t\supset t\supset t} \longmapsto 2(1\otimes\Gamma_{t\supset t\supset t})$$

$$1\otimes\Gamma_{u\supset u\supset u\supset u} \longmapsto 2(1\otimes\Gamma_{u\supset u\supset u})$$

$$1\otimes\Gamma_{s,t\supset s,t} \longmapsto -4(1\otimes\Gamma_{s,t\supset t}) + 4(1\otimes\Gamma_{s,t\supset s})$$

$$1\otimes\Gamma_{s,u\supset s,u} \longmapsto -2(1\otimes\Gamma_{s,u\supset u}) + 2(1\otimes\Gamma_{s,u\supset s})$$

$$1\otimes\Gamma_{t,u\supset t,u} \longmapsto -3(1\otimes\Gamma_{t,u\supset u}) + 3(1\otimes\Gamma_{t,u\supset t})$$

$$1\otimes\Gamma_{s,t\supset s\supset s} \longmapsto 0$$

$$1\otimes\Gamma_{s,t\supset t\supset t} \longmapsto 0$$

$$1\otimes\Gamma_{s,u\supset s\supset s} \longmapsto 0$$

$$1\otimes\Gamma_{s,u\supset u\supset u} \longmapsto 0$$

$$1\otimes\Gamma_{t,u\supset t\supset t} \longmapsto 1\otimes\Gamma_{t\supset t\supset t} - 1\otimes\Gamma_{u\supset u\supset u}$$

$$1\otimes\Gamma_{t,u\supset u\supset u} \longmapsto -1\otimes\Gamma_{u\supset u\supset u} + 1\otimes\Gamma_{t\supset t\supset t}$$

$$1\otimes\Gamma_{s,t,u\supset s} \longmapsto 2(1\otimes\Gamma_{s,t,u})$$

$$1\otimes\Gamma_{s,t,u\supset t} \longmapsto 2(1\otimes\Gamma_{s,t,u})$$

$$1\otimes\Gamma_{s,t,u\supset u} \longmapsto 2(1\otimes\Gamma_{s,t,u})$$

The $\delta_3(\Gamma_{s,t,u})$ computation is given by:

$$\begin{aligned}\delta_3(\Gamma_{s,t,u}) \quad = \quad & \sum_{\beta\in W_{\{s,t,u\}}^{\{t,u\}}} (-1)^{\ell(\beta)+1}\beta\Gamma_{t,u} + \sum_{\beta\in W_{\{s,t,u\}}^{\{s,u\}}} (-1)^{\ell(\beta)+2}\beta\Gamma_{s,u} \\ & + \sum_{\beta\in W_{\{s,t,u\}}^{\{s,t\}}} (-1)^{\ell(\beta)+3}\beta\Gamma_{s,t}\end{aligned}$$

We note here that after tensoring with $\mathbb{Z}$ each summand will become a sum of identical generators, with sign depending on the length of the minimal coset representatives. We write a short Python program, attached in Appendix A which returns the number of even and odd length minimal coset representatives, and we note that in this case for every summand the signs will cancel out.

The $\delta_4(\Gamma_{s,t,u\supset s})$ computation is given by:

$$\delta_4(\Gamma_{s,t,u\supset s}) = \sum_{i=1,2}\sum_{\tau\in\Gamma_i}\sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}}(-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma')$$

Here we compute by Python in Appendix A the coset representatives of a two generator subgroup of $\{s,t,u\}$ and the conjugates of an element of $\{s,t,u\}$ by these representatives. Whenever this conjugate is a generator of the two element subgroup, it follows that there is another coset representative which conjugates to the same generator, and these differ in length modulo 2. This means that the corresponding signs for the entries will be the opposite in the above sum and they will therefore cancel upon tensoring with $\mathbb{Z}$. A sample of this calculation is shown in Example A.2. We therefore only need to consider the case where $i=2$.

$$\begin{aligned}\delta_4(\Gamma_{s,t,u\supset s}) &= \sum_{i=2}\sum_{\tau=s}\sum_{\beta\in W_s}(-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta\Gamma_{s,t,u}\\ &= \Gamma_{s,t,u}+s\Gamma_{s,t,u}\end{aligned}$$

$$\begin{aligned}\alpha(\Gamma_{s,t,u\supset s},i,\tau,\beta) &= i\cdot\ell(\beta)+\sum_{k=1}^{i-1}|\Gamma_k|+\mu(\Gamma_i,\tau)\\ \alpha(\Gamma_{s,t,u\supset s},2,s,\beta) &= 2.\ell(\beta)+3+1\end{aligned}$$

We therefore have as generators for $H_3(W(B_3);\mathbb{Z})$:

$$\begin{aligned}\alpha &= 1\otimes\Gamma_{s\supset s\supset s}\\ \beta &= 1\otimes\Gamma_{t\supset t\supset t}\\ \gamma &= 1\otimes\Gamma_{u\supset u\supset u}\\ \delta &= 1\otimes\Gamma_{s,t\supset s}-1\otimes\Gamma_{s,t\supset t}\\ \epsilon &= \Gamma_{s,u\supset s}-1\otimes\Gamma_{s,u\supset u}\\ \eta &= \Gamma_{t,u\supset t}-1\otimes\Gamma_{t,u\supset u}\\ \iota &= 1\otimes\Gamma_{s,t,u}\end{aligned}$$

and the relations are given by the image of δ_4 as follows:

$$\begin{aligned}2\alpha=2\beta=2\gamma=4\delta=2\epsilon=3\eta=2\iota=0\\ \beta=\gamma\end{aligned}$$

So the third integral homology of $W(B_3)$ is therefore given by:

$$H_3(W(B_3);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$$

and thus we see there are no non trivial extensions for the B_3 component.

For $W(H_3)$ this gives the following resolution, where we again compute $\delta_3(\Gamma_{s,t,u})$ and $\delta_4(\Gamma_{s,t,u\supset x})$ with $x \in \{s,t,u\}$ using Python (see Appendix A):

$$\mathbb{Z} \otimes_{W_T} C_3 \xrightarrow{\delta_3} \mathbb{Z} \otimes_{W_T} C_2 \xrightarrow{\delta_2}$$

Differentials:

$$1 \otimes \Gamma_{s\supset s\supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{t\supset t\supset t} \longmapsto 0$$

$$1 \otimes \Gamma_{u\supset u\supset u} \longmapsto 0$$

$$1 \otimes \Gamma_{s,t\supset s} \longmapsto 1 \otimes \Gamma_{s\supset s} - 1 \otimes \Gamma_{t\supset t} - 2(1 \otimes \Gamma_{s,t})$$

$$1 \otimes \Gamma_{s,t\supset t} \longmapsto -1 \otimes \Gamma_{t\supset t} + 1 \otimes \Gamma_{s\supset s} - 2(1 \otimes \Gamma_{s,t})$$

$$1 \otimes \Gamma_{s,u\supset s} \longmapsto -2(1 \otimes \Gamma_{s,u})$$

$$1 \otimes \Gamma_{s,u\supset u} \longmapsto -2(1 \otimes \Gamma_{s,u})$$

$$1 \otimes \Gamma_{t,u\supset t} \longmapsto 1 \otimes \Gamma_{t\supset t} - 1 \otimes \Gamma_{u\supset u} - 2(1 \otimes \Gamma_{t,u})$$

$$1 \otimes \Gamma_{t,u\supset u} \longmapsto -1 \otimes \Gamma_{u\supset u} + 1 \otimes \Gamma_{t\supset t} - 2(1 \otimes \Gamma_{t,u})$$

$$1 \otimes \Gamma_{s,t,u} \longmapsto 0$$

$$\mathbb{Z} \otimes_W C_4 \xrightarrow{\delta_4} \mathbb{Z} \otimes_W C_3 \xrightarrow{\delta_3} \mathbb{Z} \otimes_W C_2$$

Differentials:

$$1 \otimes \Gamma_{s\supset s\supset s\supset s} \longmapsto 2(1 \otimes \Gamma_{s\supset s\supset s})$$

$$1 \otimes \Gamma_{t\supset t\supset t\supset t} \longmapsto 2(1 \otimes \Gamma_{t\supset t\supset t})$$

$$1 \otimes \Gamma_{u\supset u\supset u\supset u} \longmapsto 2(1 \otimes \Gamma_{u\supset u\supset u})$$

$$1 \otimes \Gamma_{s,t\supset s,t} \longmapsto -5(1 \otimes \Gamma_{s,t\supset t}) + 4(1 \otimes \Gamma_{s,t\supset s})$$

$$1 \otimes \Gamma_{s,u\supset s,u} \longmapsto -2(1 \otimes \Gamma_{s,u\supset u}) + 2(1 \otimes \Gamma_{s,u\supset s})$$

$$1 \otimes \Gamma_{t,u\supset t,u} \longmapsto -3(1 \otimes \Gamma_{t,u\supset u}) + 3(1 \otimes \Gamma_{t,u\supset t})$$

$$1 \otimes \Gamma_{s,t\supset s\supset s} \longmapsto 1 \otimes \Gamma_{s\supset s\supset s} - 1 \otimes \Gamma_{t\supset t\supset t}$$

$$1 \otimes \Gamma_{s,t\supset t\supset t} \longmapsto -1 \otimes \Gamma_{t\supset t\supset t} + 1 \otimes \Gamma_{s\supset s\supset s}$$

$$1 \otimes \Gamma_{s,u\supset s\supset s} \longmapsto 0$$

$$1 \otimes \Gamma_{s,u\supset u\supset u} \longmapsto 0$$

$$1 \otimes \Gamma_{t,u\supset t\supset t} \longmapsto 1 \otimes \Gamma_{t\supset t\supset t} - 1 \otimes \Gamma_{u\supset u\supset u}$$

$$1 \otimes \Gamma_{t,u\supset u\supset u} \longmapsto -1 \otimes \Gamma_{u\supset u\supset u} + 1 \otimes \Gamma_{t\supset t\supset t}$$

$$1 \otimes \Gamma_{s,t,u\supset s} \longmapsto 2(1 \otimes \Gamma_{s,t,u})$$

$$1 \otimes \Gamma_{s,t,u\supset t} \longmapsto 2(1 \otimes \Gamma_{s,t,u})$$

$$1 \otimes \Gamma_{s,t,u\supset u} \longmapsto 2(1 \otimes \Gamma_{s,t,u})$$

We therefore have as generators for $H_3(W(H_3);\mathbb{Z})$:

$$\begin{array}{rcl}
\alpha & = & 1 \otimes \Gamma_{s\supset s\supset s} \\
\beta & = & 1 \otimes \Gamma_{t\supset t\supset t} \\
\gamma & = & 1 \otimes \Gamma_{u\supset u\supset u} \\
\delta & = & 1 \otimes \Gamma_{s,t\supset s} - 1 \otimes \Gamma_{s,t\supset t} \\
\epsilon & = & \Gamma_{s,u\supset s} - 1 \otimes \Gamma_{s,u\supset u} \\
\eta & = & \Gamma_{t,u\supset t} - 1 \otimes \Gamma_{t,u\supset u} \\
\iota & = & 1 \otimes \Gamma_{s,t,u}
\end{array}$$

and the relations are given by the image of δ_4 as follows:

$$\begin{aligned}
2\alpha = 2\beta = 2\gamma = 5\delta = 2\epsilon = 3\eta = 2\iota = 0 \\
\alpha = \beta \\
\beta = \gamma
\end{aligned}$$

So the third integral homology of $W(H_3)$ is therefore given by:

$$H_3(W(H_3);\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$$

and thus we see there are no extension problems for the H_3 component.

□

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